# Stochastic approach to the receptivity problem applied to bypass transition in boundary layers

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To study the flow behavior in the presence of external disturbances of chaotic nature, a stochastic approach is pursued. In particular, transition to turbulence in boundary layers exposed to high levels of free-stream turbulence is considered. The late stages of this transition scenario, characterized by the growth and breakdown of streamwise-elongated streaks, are examined by considering the linear evolution of perturbations to a base flow consisting of the Blasius profile and the streaks. A stochastic initial condition is considered where the free-stream perturbations are described by the correlations of isotropic homogeneous turbulence. The spatial correlation of the excited flow at later times can be computed by the numerical solution of a Lyapunov equation. It is shown that free-stream turbulence has the necessary features to excite secondary energy growth, thus playing a central role in the transition to turbulence. The method proposed here can be used to examine the receptivity of other flows to external noise whose statistical properties are known or can be modeled. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2841621]

## I. INTRODUCTION

It is now well established that eigenvalue analysis alone is not sufficient to investigate the linear stability of a given flow configuration. The least stable among the eigensolutions provides relevant information only of the flow behavior at large times. Nonmodal or input-output analysis, conversely, considers also the possibility of transient energy amplifications over short times, providing, therefore, complete information on the behavior of disturbances in the flow.[1] The input-output approach is usually performed by means of optimization techniques: The initial *optimal* conditions giving the largest possible growth at a given time is sought. In the presence of a strong modal (exponential) instability, the unstable mode will be obtained from the optimization procedure, thus retrieving the results of the classic asymptotic analysis. In a stable configuration, the transient energy growth due to nonmodal mechanisms can be large enough to trigger nonlinear interactions and take the flow into a new configuration; e.g., in shear flows.[2]

The actual amplitude the perturbations attain in a physical configuration would depend on the receptivity; i.e., on the level of ambient noise and the way it affects the system. In other words, once the largest possible modal and nonmodal energy growth are known, to accurately predict the flow behavior one still needs to determine their realizability in the specific case under consideration. Here we propose a stochastic approach for computing the energy amplification that can be expected when the statistical properties of the external noise are known.

Stochastic analysis of the non-normal operators governing the linear evolution of perturbations in shear flows has been presented in previous work.[3–5] These authors identified the wavenumbers and components of the most detrimental stochastic forcing, assumed uncorrelated in space and time, as well as scaling laws for the energy growth at subcritical Reynolds numbers. Our approach here is to quantify the effect on the flow of initial perturbations whose features are known from experiments or models.

The method is applied to the case of bypass transition in boundary layers. In the presence of moderate levels of free-stream turbulence, the boundary layer flow is characterized by the nonmodal growth of streamwise-elongated low-frequency structures, called streaks.[6] When these streaks reach larger amplitudes, they become susceptible to secondary inviscid instability.[7] However, it has been shown[8,9] that the streaks can break down already at subcritical amplitudes for the transient growth of high-frequency perturbations. A transition scenario, a sequence of several transient-growth mechanisms, has been therefore suggested (see also Ref. 10). The optimal initial condition riding on the streak consists of velocity perturbations localized in the regions of highest shear of the base flow and tilted upstream of the flow.[9] Owing to the specific shape of this initial condition, the aim of this article is to investigate whether the mechanisms of transient growth on the streak are excited in a flow subject to free-stream turbulence and what energy growth compared to the optimal can be expected.

## II. PROBLEM FORMULATION

The dynamical system governing the evolution of the perturbation we wish to examine is given by the Navier–Stokes equations linearized around a parallel streaky base flow $(U,V,W)=[U(y,z),0,0]$, where the velocity compo-

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nents are relative to the streamwise ($x$), wall-normal ($y$), and spanwise ($z$) directions, respectively. In the analysis, the streaks resulting from the nonlinear evolution of the spatial optimal perturbation in a Blasius boundary layer are considered.[11] We are interested in determining the local properties of the streaks in the parallel flow approximation since we consider a flow evolving slowly in the streamwise direction and perturbations evolving faster than the base flow. This approximation therefore becomes questionable for perturbations of long streamwise scale or when the behavior at large times is examined. The streak amplitudes under consideration range between 20% and 30% of the free-stream velocity; i.e., stable or marginally unstable streaks. For further details on the base flow, see Ref. 9 and references therein. The equations governing the linear evolution of a perturbation velocity field can be reduced to 2 by using the normal velocity $v$ and the normal vorticity $\eta = u_z - w_x$ (see, e.g., Ref. 2),

$$\Delta v_t + U\Delta v_x + U_{zz}v_x + 2U_z v_{xz} - U_{yy}v_x - 2U_z w_{xy} - 2U_{yz}w_x = \frac{1}{\mathrm{Re}}\Delta\Delta v, \quad (1)$$

$$\eta_t + U\eta_x - U_z v_y + U_{yz}v + U_y v_z + U_{zz}w = \frac{1}{\mathrm{Re}}\Delta\eta.$$

In the above, the spanwise velocity $w$ can be eliminated by using the identity

$$w_{xx} + w_{zz} = -\eta_x - v_{yz}.$$

Since the flow is assumed parallel, solution can be sought in the form of normal modes

$$[v,\eta] = [\hat{v}(y,z,t), \hat{\eta}(y,z,t)]e^{ik_x x} + \text{c.c.}, \quad (2)$$

where $k_x$ is the streamwise wavenumber. As the basic flow is symmetric about $z=0$, the modes can be further divided into separate classes according to their odd or even symmetry with respect to the basic flow. Details on the numerical methods used to discretize Eqs. (1) can be found in Ref. 9.

## III. STOCHASTIC APPROACH

In a stochastic framework, the flow field is described by its correlation matrix containing the second-order statistics of the velocity components. In this work, a deterministic system with stochastic inputs is considered. A discrete formulation is presented here; the corresponding definitions pertaining to continuous differential operators can be found, e.g., in Refs. 12 and 13. The dynamical system under consideration is written as $\dot{q} = Aq$, $q(0) = q_0$, where $q$ consists of the wall-normal velocity and vorticity perturbation. The covariance of $q$ at time $t$ is defined as $P(x,x',t) = \mathrm{cov}[q(x,t), q(x',t)] = \langle qq^H \rangle$, where $x$ indicates the independent spatial variables, $\langle\,\rangle$ ensemble averaging, and superscript $H$ the conjugate transpose. In order to extract physically relevant information from the covariance matrix, it is useful to associate an inner product with its definition. If we are interested in the perturbation kinetic energy $E$, a weighted inner product can be introduced in such a way that $E = q^H Q q$, where $Q$ is a matrix containing integration weights pertaining to the specific discretization scheme. The weighted covariance matrix then becomes $\hat{P} = PQ$. Different flow quantities can be extracted from $\hat{P}$: The trace of it provides the flow mean kinetic energy, whereas its eigenvectors and eigenvalues are the most energetic structures in the flow [proper orthogonal decomposition (POD) modes]. Indeed, the eigenvectors of $\hat{P}$ yield orthonormal basis vectors for the kinetic energy inner product. In addition, the eigenvalues provide the mean kinetic energy of the coherent processes associated to each mode since the corresponding expansion coefficients are uncorrelated.

A Lyapunov equation can be obtained for the evolution of the covariance of the state

$$\dot{P} = AP + PA^H, \quad P(0) = P_0. \quad (3)$$

This equation represents an initial value problem for a flow whose state at time $t=0$ is known in a statistical sense. Note that it is independent of the choice of a relevant inner product and therefore the covariance for the desired norm can be extracted afterwards. The solution to this problem can be written by means of the exponential matrix

$$P(t) = e^{At}P_0 e^{A^H t}. \quad (4)$$

For large matrices, when the solution of the Lyapunov equation can become a computationally hard task, the covariance of the state can be computed by integration of the POD modes $\phi_i$ of $P_0$ to time $t$, $\phi_i(t)$. The covariance $P(t)$ is then reconstructed using the eigenvalues $\lambda_i$ of $P$ as weights; i.e., $P(t) = \Sigma_i \phi_i(t)^H \lambda_i \phi_i(t)$. In this case, it suffices to integrate $n$ initial conditions forward in time, provided the energy associated to the $n+1$-mode is small enough to be neglected.

## IV. COVARIANCE OF THE EXTERNAL DISTURBANCES

For the results presented here, the covariance of the initial perturbation $P_0$ is defined by assuming homogeneous isotropic turbulence in the free stream. The turbulence spectrum is modeled by the von Kármán spectrum, which is proportional to $\kappa^4$ for large scales and matches the Kolmogorov (5/3)-law for small scales. It has the form

$$\tilde{E}^{3\mathrm{D}}(\alpha,\beta,\gamma) = \frac{2}{3}\frac{1}{4\pi\kappa_{3\mathrm{D}}^2}\frac{1.606(\kappa_{3\mathrm{D}}L)^4}{(1.35+(\kappa_{3\mathrm{D}}L)^2)^{17/6}}Lq, \quad (5)$$

where the wave vector $\kappa_{3\mathrm{D}}^2 = k_x^2 + k_y^2 + k_z^2$, $L$ is an integral length scale, and $q$ is the total turbulent kinetic energy, defined as the integral over all $\kappa$ of the spectrum. It is possible to show that the integral length scale $L_{11} \approx 0.65L$ and that the length scale of the eddies with the maximum energy is $L_{\max} \approx 3.5L$.[14] The Fourier transform of the velocity correlations can be computed directly from the spectrum as

$$\langle u_i u_j \rangle = \tilde{E}^{3\mathrm{D}} \cdot (\delta_{ij} - k_i k_j/\kappa_{3\mathrm{D}}^2).$$

Since the governing equations are solved in spectral space in the homogeneous streamwise and periodic spanwise directions ($x$ and $z$, respectively), and in physical space in the wall-normal direction $y$, an inverse Fourier transform is ap-

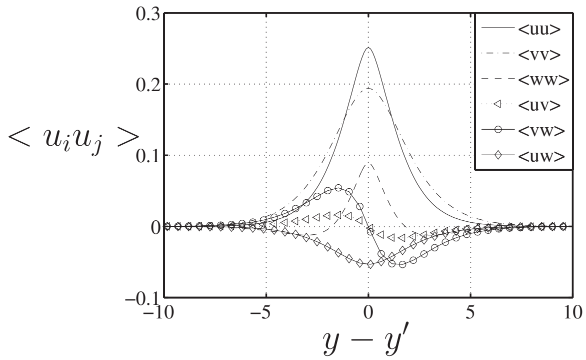


FIG. 1. Independent components of the symmetric covariance matrix $R_{ij}=\langle u_i u_j\rangle(k_x,k_z,y-y')$ for homogeneous isotropic fluctuations with $k_x=0.3$, $k_z=1$, and turbulence length scale $L=10$.

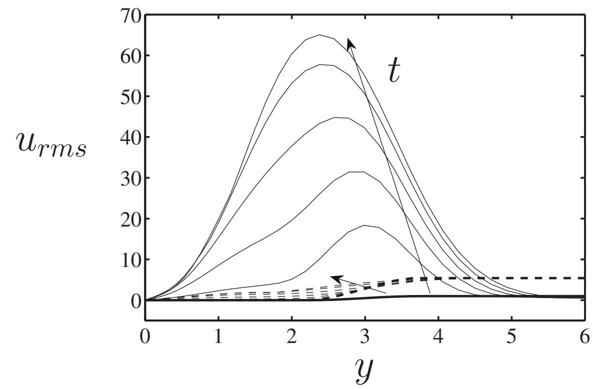


FIG. 3. Evolution of the wall-normal profiles of the streamwise (solid line) and spanwise (dashed line) velocity fluctuations. Sinuous antisymmetric perturbation with $k_x=0.05$, $\mathrm{Re}_{\delta*}=1000$, and streak amplitude $A=0.25$.

plied in $k_y$ to retrieve $R_{ij}=\langle u_i u_j\rangle(k_x,k_z,y-y')$. As an example, Fig. 1 displays the six independent components of the symmetric covariance matrix $R_{ij}$ for the case of $k_x=0.3$, $k_z=1$, and $L=10$. Note that the $\langle uw\rangle$ Reynolds stress is not zero for $y=y'$ when single wavenumber pairs $(k_x,k_z)$ are considered; it indeed becomes zero upon integration in space as expected for homogeneous isotropic turbulence. The covariance matrix thus obtained can be reformulated in terms of the wall-normal velocity and vorticity exploiting the properties of linear transformations as $P_0=CR_{ij}C^H$, where $C$ maps the three velocity components $(u,v,w)$ into $(v,\eta)$. To consider the mechanisms of penetration of perturbations from the free stream, the turbulence is assumed to be initially present only in the outer part of the boundary layer, outside the shear region close to the wall. The initial covariance matrix $P_0$ is adjusted so that the perturbation fluctuations are damped to zero close to the wall. This introduces spurious anisotropy at the edge of the boundary layer, similar to those characterizing the continuous modes of the stability operator.

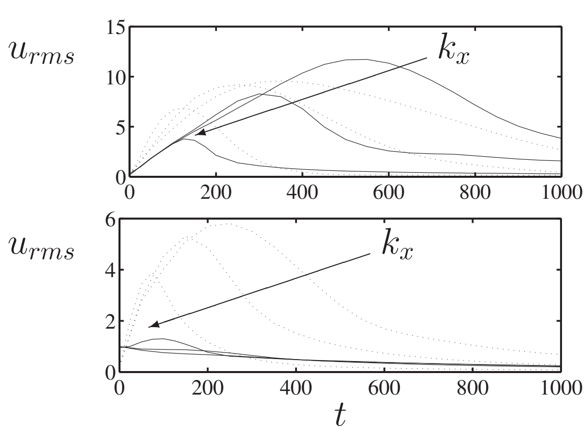


FIG. 2. Time evolution of the velocity fluctuations in the case of stochastic noise in the free stream [$u_{rms}$: (—)] and for the optimal initial conditions giving the maximum possible growth [$1/5\sqrt{E/E_0}$: $(\cdots\cdots)$] for $k_x=0.05$, 0,1, and 0.3, $\mathrm{Re}_{\delta*}=1000$, and streak amplitude $A=0.25$. (a) Sinuous antisymmetric perturbation and (b) symmetric varicose modes. The rms values from the stochastic analysis are normalized with the initial turbulence intensity $Tu=\sqrt{(u_{\mathrm{rms}}^2+v_{\mathrm{rms}}^2+w_{\mathrm{rms}}^2)}$.

## V. RESULTS

The level of disturbance amplification that can be expected in the presence of free-stream turbulence is considered first. Figure 2 shows the time evolution of the velocity fluctuations (rms values) of antisymmetric sinuous and symmetric varicose perturbations of different streamwise wavelength evolving on a linearly stable streaky base flow. The envelope of the optimal initial conditions giving maximum growth at any time $t$ is reported for comparison.[9] (Note that the values are divided by 5 to improve the plot readability.) The results indicate that, in the case of stochastic noise in the free stream, sinuous perturbations grow about 20% of the optimal deterministic perturbation, whereas the varicose modes experience negligible growth, if any. Only the streamwise velocity fluctuations are reported in the figure, the other components being just decaying and of smaller amplitude. This quantifies the realizability of the optimal energy growth computed in Ref. 9. These authors found the optimal initial conditions to be localised in the regions of highest shear of the streak and to be tilted upstream; however in the case of more realistic ambient noise a lower but still significant growth is expected only for sinuous perturbations. This can be explained by the fact that free-stream sinuous modes are characterized by largest fluctuations in the spanwise velocity component, the component responsible for the observed transient growth,[9] whereas symmetric perturbations have strong contributions from the streamwise and wall-normal velocity and are therefore less efficient in exciting nonmodal growth. Note also that the maximum amplification is reached at later time in the case of initial free-stream disturbances.

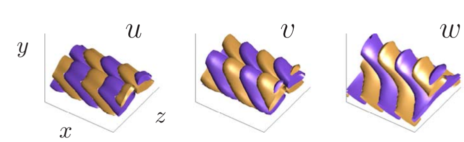


FIG. 4. (Color online) The most energetic flow structure, first POD mode, at the time of maximum growth. $k_x=0.3$, $\mathrm{Re}_{\delta*}=1000$, and streak amplitude $A=0.25$.

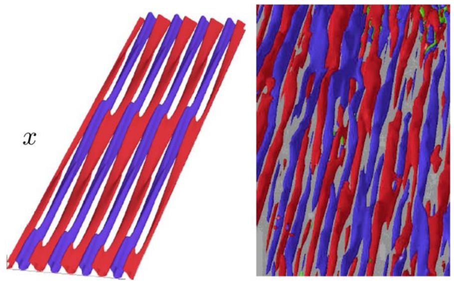


FIG. 5. (Color online) Visualizations of the streaky pattern from the numerical simulations of a boundary layer exposed to free-stream turbulence (performed by P. Schlatter) and from the present results with streak amplitude $A=0.25$ and secondary perturbations of $k_x=0.05$. The flow is from bottom to top.

To follow the penetration of the free-stream perturbation in the boundary layer the time evolution of the wall-normal profiles of the streamwise and spanwise velocity fluctuations is reported in Fig. 3 for the case of free-stream modes present above $y=3$. The most detrimental initial condition consists mainly of the spanwise velocity component (see also Ref. 8), whereas the largest flow response is in the streamwise component. Indeed, the figure shows that the spanwise velocity perturbation is slowly diffusing towards the wall while the streamwise velocity is experiencing a significant growth.

Figure 4 displays the first POD mode of the covariance matrix $P$ at the time of maximum growth for the case of antisymmetric disturbances with $k_x=0.3$. This mode represents the most energetic flow structure. The perturbation is very similar to that obtained from the evolution of the deterministic optimal initial condition[9] as well as to the shape of the unstable sinuous modes,[7] so that similar breakdown scenarios can be expected independently of the type of triggering mechanism. Note also that the structure of the flow response is almost independent of the streamwise wavelength of the disturbances. The POD modes of the initial free-stream turbulence consist of sinusoidal waves in the wall-normal direction, the most energetic being the modes with lower wall-normal wavenumber.

Finally, we would like to compare the flow structures from the numerical simulations of a boundary layer exposed to free-stream turbulence[14] to those obtained by superimposing the flow response to stochastic free-stream disturbances on the underlying streaks. As shown in Fig. 5, thinning and bending of the upper part of the streak can be seen in both cases.

## VI. CONCLUSIONS

A stochastic approach for studying the effect of free-stream turbulence on boundary layer streaks is presented. The results indicate that free-stream turbulence has the necessary features to excite significant secondary energy growth of antisymmetric high-frequency perturbations, thus playing a central role in the transition to turbulence. The results are in agreement with the simulations in Ref. 15, where it is shown that one single free-stream mode is able to trigger the streak breakdown. The analysis performed also indicates that the streaks are most sensitive to low-wavenumber perturbations, as shown in Ref. 9. These disturbances are responsible for the observed slow streak meandering, which may trigger breakdown to turbulence by inducing interactions among streaks.[14,16] The method proposed here can be used to examine the receptivity of other flows to external noise whose statistical properties are known or can be modeled. The approach allows improvement of predictions of the flow behavior based only on optimal growth theory. The exact optimal perturbations are seldom encountered in practical configurations (see, e.g., the case of the primary transient growth of streaks).[17]

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