Circulating electrons, superconductivity, and the Darwin-Breit interaction

Hanno Essén
Department of Mechanics
Royal Institute of Technology
S-100 44 Stockholm, Sweden

August 1999

Abstract

The importance of the Darwin-Breit interaction between electrons in solids at low temperatures is investigated. The model problem of particles on a circle is used and applied to mesoscopic metal rings in their normal state. The London moment formula for a rotating superconducting body is used to calculate the number, N, of superconducting electrons in the body. This number is found to be equal to the size, R, of the system divided by the classical electron radius, i.e. $N = Rmc^2/e^2$. The Darwin-Breit interaction gives a natural explanation for this relation from first principles. It also is capable of electron pairing. Collective effects of this interaction require a minimum of two dimensions but electron pairing is enhanced in one-dimensional systems.

Unpublished manuscript.
Available from: http://xxx.lanl.gov/abs/cond-mat/0002096

1 Introduction

Arguments and results will be presented that hopefully convince the open-minded reader that superconductivity is caused by the Darwin-Breit (magnetic) interaction between semiclassical electrons. The starting point is a careful study of the model problem of electrons on a circle. This simple model is chosen since it allows accurate treatment of the notoriously difficult problem of relativistic and magnetic effects in many-electron systems. Since classical ideas are closer to our intuition the classical picture is taken as far as possible before quantum mechanics is reluctantly adopted. The semiclassical point of view is an extremely powerful one [1, 2] and the reader will find further examples of this below.

Relativistic quantities, to a first approximation, have a magnitude $(v/c)^2$ times those of non-relativistic quantities. While this always is small in everyday life, in the atomic world this parameter is $\sim 10^{-4}$, which is fairly small, but rarely negligible. A striking example of this is the energy gap in superconductors which typically is order of magnitude 10^{-4} of the Fermi energy. Any study of this phenomenon that does not take relativistic effects into account must consequently remain inconclusive

The Darwin-Breit interaction [3, 4, 5] is the first order relativistic correction,

$$V_1 = -\sum_{i<j}^{N} \frac{e^2}{c^2} \frac{\boldsymbol{v}_i \cdot \boldsymbol{v}_j + (\boldsymbol{v}_i \cdot \boldsymbol{e}_{ij})(\boldsymbol{v}_j \cdot \boldsymbol{e}_{ij})}{2r_{ij}}, \tag{1}$$

to the Coulomb potential. Sucher [6] in a recent review (*What is the force between two electrons?*) gives a thorough discussion of its origin in QED. While well known as an important perturbation in accurate atomic calculations [7, 8] it has until recently (Essén [9, 10, 11, 12]) usually been taken for granted, without proof or justification, that it is negligible in larger systems. Welker [13] suggested in 1939 that magnetic attraction of parallel currents might cause superconductivity, but after that the idea seems to have been forgotten. Other types of magnetic interaction have been suggested though [14]. Some efforts to include the Darwin-Breit interaction in density functional approaches to solids are reviewed in Strange [7]. Capelle and Gross [15] have also made efforts towards a relativistic theory of superconductivity.

In section 2 we introduce the analytical mechanics of particles on a circle and apply it to mesoscopic rings. This serves to introduce the mathematical model and also throws some light of the theory behind the persistent currents found in these. We later find that, though these rings are not superconducting, electron pairing might be relevant to understand their physics.

Section 3 closes in upon the main subject of superconductivity. The London moment formula connecting the angular velocity of a superconducting body and the magnetic field it produces is introduced and motivated. The formula, together with classical electromagnetism can be used to calculate the number of superconducting electrons present. This number is found to be determined entirely by fundamental constants and the size of the body.

Finally in section 4 the importance of the Darwin-Breit interaction is investigated. We show how it can lead to electron pairing and calculate the relevant temperatures at which these form. We also investigate when the interaction might become dominating and find that exactly the combination of number, size, and fundamental constants that followed from the London moment is the condition for this. When the condition is fulfilled the particles no longer move individually, or in pairs, but collectively. The behavior of this condition as a function of spatial dimension is investigated. Interestingly it is found that the one-dimensionality of the ring enhances pair-formation but suppresses collective behavior (superconductivity). After that the conclusions are summarized.

2 Rings, persistent currents, and flux periodicity

In solid state physics cold mesoscopic metal rings have attracted a lot of attention. In particular since theoretical predictions [16, 17] that an external magnetic flux through the ring causes a persistent current round it, have been experimentally verified [18, 19, 20]. The agreement between theory and experiment is, however, still far from perfect [21], for reviews see [22, 23]. One normally assumes that it is correct to treat the conduction electrons semiclassically, one speaks about ballistic electrons [1, 23], and we will do so here. Superconductivity is not treated in this section, but we assume that the rings are perfect conductors (have zero resistance).

2.1 Charged particles on a circle

We now set up the model problem of charged particles constrained to move on a circle. Assuming that the circle has radius R, positions and velocities are given by

$$\boldsymbol{r}_i(\varphi_i) = R\boldsymbol{e}_\rho(\varphi_i), \quad \text{and} \quad \boldsymbol{v}_i(\varphi_i, \dot{\varphi}_i) = R\dot{\varphi}_i \boldsymbol{e}_\varphi(\varphi_i), \tag{2}$$

where $\boldsymbol{e}_\rho(\varphi) = \cos\varphi\, \boldsymbol{e}_x + \sin\varphi\, \boldsymbol{e}_y$ and $\dot{\boldsymbol{e}}_\rho = \dot{\varphi}\boldsymbol{e}_\varphi$, as usual. We take the zeroth order Lagrangian to be

$$L_0 = T_0 - V_0 = \frac{1}{2}\sum_{i=1}^{N} m_i R^2 \dot{\varphi}_i^2 - V_0(\varphi_1, \ldots, \varphi_N). \tag{3}$$

Since we will have metallic conduction electrons in mind the potential V_0 does not necessarily represent the Coulomb interactions, but rather interactions with the lattice plus, possibly, Debye screened two particle interactions. The generalized (angular) momenta are $J_i = \partial L_0/\partial\dot{\varphi}_i = mR^2\dot{\varphi}_i$ so the Hamiltonian is

$$H_0 = \sum_{i=1}^{N} \frac{J_i^2}{2m_i R^2} + V_0. \tag{4}$$

If there is a magnetic flux $\Phi = \int \boldsymbol{B}\cdot \mathrm{d}\boldsymbol{s} = \oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r} = 2\pi R A_\varphi$ through the ring the Hamiltonian changes to

$$H_0 = \sum_{i=1}^{N} \frac{1}{2m_i}\left(\frac{J_i}{R} - \frac{e_i}{c}A_\varphi\right)^2 + V_0 = \sum_{i=1}^{N} \frac{1}{2m_i R^2}\left(J_i - \frac{e_i}{2\pi c}\Phi\right)^2 + V_0, \tag{5}$$

since $A_\varphi = \Phi/(2\pi R)$.

We find the equations of motion

$$\dot{J}_i = -\frac{\partial H_0}{\partial \varphi_i} = -\frac{\partial V_0}{\partial \varphi_i}, \tag{6}$$

$$\dot{\varphi}_i = \frac{\partial H_0}{\partial J_i} = \frac{J_i}{m_i R^2} - \frac{e_i \Phi}{m_i R^2 2\pi c}. \tag{7}$$

The current round the ring is by definition

$$I = \sum_{i=1}^{N} e_i \frac{\dot{\varphi}_i}{2\pi} = \frac{1}{2\pi} \sum_{i=1}^{N} \left(\frac{e_i J_i}{m_i R^2} - \frac{e_i^2 \Phi}{m_i R^2 2\pi c} \right) \equiv I_0 + I_\Phi. \tag{8}$$

One notes that the relation

$$I = -c \frac{\partial H_0}{\partial \Phi} \tag{9}$$

holds.

For non-interacting particles on the ring we have

$$V_0 = \sum_{i=1}^{N} U_0(\varphi_i). \tag{10}$$

Then $H_0 = \sum_i H_i(J_i, \varphi_i)$ where H_i are constants of the motion, $H_i = E_i$, whether there is a flux or not. There are then the adiabatic invariants [24]

$$I_{\varphi_i} \equiv \frac{1}{2\pi} \oint J_i(\varphi_i; E_i, \Phi) \mathrm{d}\varphi_i = \overline{J_i}, \tag{11}$$

the averages, $\overline{J_i}$, of the J_i round the ring. If the flux is turned on slowly they will retain their zero flux values. The zero flux average current

$$\overline{I_0} = \frac{1}{2\pi R^2} \sum_{i=1}^{N} \frac{e_i \overline{J_i}}{m_i} \tag{12}$$

is thus also an adiabatic invariant, and remains constant. This means that slowly turning on a flux Φ through the ring results in the extra diamagnetic circulating current

$$I_\Phi = -\frac{\Phi}{4\pi^2 R^2} \sum_{i=1}^{N} \frac{e_i^2}{m_i c} \tag{13}$$

independently of any pre-existing current. Below we will find that the above result can be found using Larmor's theorem and thus, in fact, is independent of electron interactions provided other conditions are fulfilled.

2.2 Two types of current

We find that there are two different types of current possible in these rings. The 'ballistic' current I_0, which should be, at most [25], order of magnitude a few

$ev_{\rm F}/(2\pi R)$, where $v_{\rm F}$ is the Fermi velocity, and the Larmor current I_Φ induced by the flux. Assuming that only electrons contribute (13) becomes

$$I_\Phi = \frac{\Phi}{4\pi^2 R^2} N \frac{e^2}{mc}. \tag{14}$$

Putting

$$\Phi = n_\phi \frac{hc}{|e|} \equiv n_\phi \Phi_0, \tag{15}$$

where n_ϕ is dimensionless and $\Phi_0 = hc/|e|$ is the flux quantum, we get the expression $I_\Phi \pi R^2 = -N n_\phi \mu_{\rm B}$. Here $\mu_{\rm B} = |e|\hbar/(2m)$ is the Bohr magneton. Gaussian units are used in most formulas; to get equation (14) in SI-units we simply delete c. If the flux is $\Phi = B\pi R^2$ we can then rewrite it in the form

$$I_\Phi = -N \cdot B \cdot 2.242\,\mathrm{nA/T}. \tag{16}$$

To get a number out of this formula we must estimate the number N of semiclassical electrons and know the magnetic field in teslas. The speed corresponding to the Larmor current is, in atomic units, $v_\Phi = n_\phi/R \ll v_{\rm F} = 1.92/r_s$, where r_s is the radius parameter. On the other hand all semiclassical electrons contribute to I_Φ, whereas the number contributing to I_0 necessarily is small.

Levy et al. [19] found an average current of $I_{\rm av} = 3 \cdot 10^{-3} \cdot ev_{\rm F}/\ell = 0.36\,\mathrm{nA}$ in their Cu-rings, of circumference $\ell = 2.2\,\mu$m. If this is interpreted as a Larmor-current we can calculate N. At the magnetic field $B_0 = 1.3 \cdot 10^{-2}\,\mathrm{T}$ corresponding to the flux quantum Φ_0 this gives the reasonable result $N \approx 100$ for the number of semiclassical electrons in the system. Chandrasekhar et al. [20], on the other hand, found currents $I =(0.3 - 2.0)\; ev_{\rm F}/(2\pi R)$ in a single gold ring. These can thus only be interpreted as due to ballistic currents. They might be due to electron pairs, which may form even in the normal state, as we will see below.

2.3 Larmor's theorem

Consider a system of particles, all of the same charge to mass ratio e/m. Assume that they move in a common external potential, $U_e(\rho, z)$, that is axially symmetric, i.e. independent of φ, under the influence of arbitrary interparticle interactions. Now place this system in a weak magnetic field, B_z, along the z-axis. One can then apply Larmor's theorem [7, 26] to show that the response of the system to this field is a rotation with angular velocity

$$\Omega_z = -\frac{e}{2mc} B_z \tag{17}$$

given by the Larmor frequency.

This means that there will be a circulating Larmor current

$$I_L = Ne\frac{\Omega_z}{2\pi} = -\frac{B_z}{4\pi} N \frac{e^2}{mc} \tag{18}$$

where Ne is the total amount of charge on the particles (N is not necessarily the number of particles). If we insert $B_z = \Phi/(\pi R^2)$ we recover essentially equation (14). This is why we called I_Φ the Larmor current. Note that we derived (14) under the assumption of arbitrary charge to mass ratios e_i/m_i but no interparticle interaction. Here we need identical charge to mass ratios e/m and an axially symmetric external field but can have arbitrary interactions between the particles. The general results (14) and (16) for semiclassical electrons (or electron pairs or groups) in cold metal rings thus seem fairly reliable.

It is noteworthy that the result of equation (13) is not necessarily due to any magnetic field affecting the particles. The flux Φ could very well go through a smaller surface completely inside the ring material. This means that the current in (13) is a classical Aharonov-Bohm effect [27]. That is, an effect due to the vector potential at zero magnetic field. By contrast the Larmor result (18) is derived assuming that the magnetic field penetrates the ring.

2.4 Quantizing the electron on the circle and flux periodicity

The above results are purely classical. When we quantize them we will find that physical properties must be periodic in (half?) the flux quantum, as will now be shown show. Our previous classical results for currents must be thought of as averages over these quantum periods (beats). Flux quantization was originally suggested by London [28], for a thorough discussion see Thouless [29].

The classical Hamiltonian of an electron moving freely on a circle of radius R threaded by a flux Φ is, according to equation (5),

$$H = \frac{1}{2mR^2}\left(J + \frac{|e|}{2\pi c}\Phi\right)^2. \tag{19}$$

We quantize this by letting $J \to \hat{J} = -\mathrm{i}\hbar\partial/\partial\varphi$ and thus get the Schrödinger equation

$$\frac{\hbar^2}{2mR^2}\left(-\mathrm{i}\frac{\partial}{\partial\varphi} + n_\phi\right)^2 \psi(\varphi) = E\psi(\varphi), \tag{20}$$

where we have used equation (15). Putting

$$\psi(\varphi) = \exp(-\mathrm{i}n_\phi\varphi)\,\psi'(\varphi) \tag{21}$$

we get

$$-\frac{\hbar^2}{2mR^2}\frac{\partial^2}{\partial\varphi^2}\psi' = E\psi' \tag{22}$$

for the gauge transformed wave function. It is now frequently argued [30, 17] that the wave function must be single valued and that therefore

$$\psi(\varphi + 2\pi) = \psi(\varphi). \tag{23}$$

Via (21) this leads to the physical condition

$$\psi'(\varphi + 2\pi) = \exp(\mathrm{i}n_\phi 2\pi)\psi'(\varphi) \tag{24}$$

on the solutions of (22), where the flux has been transformed away. This boundary condition is unchanged if n_ϕ changes by unity. This implies that physical quantities must be periodic in the flux with period Φ_0.

The above argument is not necessarily reliable, however. The correct wave function for an electron is a spinor (in the non-relativistic case a two component spinor). A spinor is well known to change sign when rotated by 2π. The question is then: will the spinor rotate as the electron travels round the circle? A free electron is known to have conserved helicity, the projection of the spin on the momentum. As the ring radius is large compared to atomic dimensions the electron momentum turns slowly and it seems reasonable that the helicity will remain conserved (as an adiabatic invariant). This, of course, means that the spinor must rotate with the momentum. The conclusion of all this is that the correct condition on the spinor wave function, for a single electron, should be

$$\psi(\varphi + 4\pi) = \psi(\varphi), \tag{25}$$

and thus that

$$\psi'(\varphi + 4\pi) = \exp(\mathrm{i}n_\phi 4\pi)\psi'(\varphi). \tag{26}$$

This condition is unchanged whenever n_ϕ changes by one half. I.e. physical quantities must be periodic in the flux with period $\Phi_0/2$. Note that the same result is obtained if $|e|$ in equation (19) is changed to $2|e|$. The n_ϕ in (20) changes to $2n_\phi$ and equation (24) becomes identical to (26).

In conclusion the observation of the $\Phi_0/2$ periodicity does not necessarily imply electron pairs. It might be due to single electrons going round the ring with conserved helicity. Both the Φ_0 and the $\Phi_0/2$ periodicities have been experimentally observed [31, 32, 33, 18, 19, 20].

3 Rotating superconductors and the number of superconducting electrons

There is another surprising result concerning circulating electrons that is easily explained by Larmor's theorem (17). London [28] showed (see also [34, 35, 36]), using his phenomenological theory of superconductivity, that a superconducting sphere that rotates with angular velocity $\boldsymbol{\Omega}$ will have an induced magnetic field (Gaussian units)

$$\boldsymbol{B} = \frac{2mc}{|e|}\boldsymbol{\Omega} \tag{27}$$

in its interior. Here m and e are the mass and charge of the electron. This prediction has been experimentally verified with considerable accuracy and is equally true for high temperature and heavy fermion superconductors [37, 38]. With minor modifications it is also valid for other axially symmetric shapes of the body, for example cylinders or rings.

3.1 Understanding the London moment

The London field, or 'moment', (27) can be thought of as follows. Assume that the superconducting body can be viewed as a system of interacting particles with the electronic charge to mass ratio confined by an axially symmetric external potential. When the body rotates we can transform the equations of motion to a co-rotating system, in which it is at rest, but in this system the particles will be affected by a Coriolis force $-m2\boldsymbol{\Omega}\times\boldsymbol{v}$. Larmor's theorem teaches us that such a Coriolis force is equivalent to an external magnetic field. Magnetic fields are, however, not allowed inside superconductors according to the Meissner effect. To get rid of the Coriolis forces the rotation induces surface supercurrents that produce a suitable compensating magnetic field $\boldsymbol{B}$. The Lorentz force of this field is $-(|e|/c)\boldsymbol{v}\times\boldsymbol{B}$. Provided the relation between $\boldsymbol{B}$ and $\boldsymbol{\Omega}$ is given by (27) the two forces cancel. The equations of motion in the rotating system are then the same, in the interior, as if the system did not rotate. The disturbance from the rotation on the dynamics is minimized.

The above explanation may sound compelling, but the most direct way of understanding formula (27) is, in fact, much simpler. The superconducting electrons, which are always found just inside the surface [28], are not dragged by the positive ion lattice so when it starts to rotate the superconducting electrons ignore this and remain in whatever motion they prefer. This, however, means that there will be an uncompensated motion of positive charge density on the surface of the body. This surface charge density, σ, will, of course, be the same as the density of superconducting electrons, but of opposite sign, and will produce the magnetic field. Using this we can calculate the number, N, of superconducting electrons.

3.2 The number of superconducting electrons

It is well known that a rotating uniform surface charge density will produce a uniform interior magnetic field in a sphere. If this rotating surface charge density is σ, then the total charge Q is given by

$$Q = N|e| = 4\pi R^2\sigma, \tag{28}$$

and the resulting magnetic field in the interior is

$$\boldsymbol{B} = \frac{2}{3}\frac{Q}{cR}\boldsymbol{\Omega} = \frac{8\pi}{3}\frac{\sigma R}{c}\boldsymbol{\Omega}, \tag{29}$$

where R is the radius of the sphere (relevant formulas for the calculation can be found in Essén [26]). Putting $Q = N|e|$ and comparing this equation with

(27) one finds that the number N must be given by $N = Rmc^2/e^2 = R/r_\mathrm{e}$. We thus find that the relationship

$$\frac{Nr_\mathrm{e}}{R} = 1, \tag{30}$$

where r_e is the classical electron radius, and N the number of electrons contributing to the supercurrent, characterizes the superconductivity on a sphere of radius R.

The corresponding calculation for a cylinder, long enough for edge effects to be negligible, is elementary and gives $Nr_\mathrm{e}/\ell = 1$, where ℓ is the length of the cylinder. We will return to the crucial significance of the dimensionless combination Nr_e/R below. It is noteworthy that the number N depends only on the geometry (size) and fundamental constants (r_e). How can this be if superconductivity is caused by some effective interaction with the lattice?

4 Pairing and collective effects due to the Darwin-Breit interaction

We now continue the study of the semiclassical (ballistic) electrons in the ring using the model of charged particles constrained to move on a circle. Now we further assume that the electrons are free particles to zeroth order and investigate how this is affected by the first order Darwin-Breit term. The relativistic mass-velocity correction is probably not of much interest here.

4.1 The Darwin-Breit term on the ring

For the positions and velocities of equation (2) the Darwin-Breit term (1) becomes

$$V_1 = -\frac{e^2}{Rc^2}\sum_{i<j}^{N} R^2\dot\varphi_i\dot\varphi_j \frac{1}{4}\frac{1+3\cos(\varphi_i-\varphi_j)}{\sqrt{2[1-\cos(\varphi_i-\varphi_j)]}} \equiv -\frac{e^2R}{c^2}\sum_{i<j}^{N}\dot\varphi_i\dot\varphi_j V_\varphi(\varphi_i-\varphi_j), \tag{31}$$

and the first order Lagrangian $L = T_0 - V_1$, with T_0 given in equation (3), is

$$L = \frac{1}{2}mR^2\sum_{i=1}^{N}\dot\varphi_i^2 + \frac{e^2R}{c^2}\sum_{i<j}\dot\varphi_i\dot\varphi_j V_\varphi(\varphi_i-\varphi_j). \tag{32}$$

The nature of the function V_φ is indicated in equation (42) below. If we introduce (note that the electron has charge $e = -|e|$)

$$A_i = \frac{e}{c}\sum_{j(\neq i)}^{N}\dot\varphi_j V_\varphi(\varphi_i-\varphi_j) \tag{33}$$

we can write this

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m R^2 \dot{\varphi}_i^2 + \frac{e}{2c} R \dot{\varphi}_i A_i \right). \tag{34}$$

It is easy to show that for real electrons distributed round a (one-dimensional) ring of real atoms the Darwin-Breit term will always be a small perturbation [25]. Individual terms in the interaction may still be large if some pair of interparticle distances is very small. This would correspond to pair formation and is treated in the next subsection. In the real world of two and three dimensions the Darwin-Breit term as a whole can become large. This means that individually moving particles is no longer a good first approximation. This is shown in the following subsection.

4.2 The one-dimensional hydrogen atom

The Darwin-Breit term represents an interaction which is attractive for parallel currents. For small relative velocities of the electrons it seems possible that it could lead to bound states (for the relative motion of the particles). Let us investigate this. Most conduction electrons in the metal ring will be inside the (one-dimensional) Fermi surface and they will occur in pairs of opposite momentum with no net current. Assume that only two electrons have unpaired momenta and move in the same direction around the ring approximately with the Fermi velocity. The Lagrangian of these two is then

$$L = \frac{mR^2}{2} (\dot{\varphi}_1^2 + \dot{\varphi}_2^2) + \frac{e^2 R}{c^2} \dot{\varphi}_1 \dot{\varphi}_2 V_\varphi(\varphi_1 - \varphi_2). \tag{35}$$

We now make the coordinate transformation

$$\varphi_{\mathrm{C}} = \frac{1}{2}(\varphi_1 + \varphi_2), \qquad \varphi = \varphi_1 - \varphi_2 \tag{36}$$

to center of mass angle φ_{C} and relative angle φ. The inverse transformation is

$$\varphi_1 = \varphi_{\mathrm{C}} + \frac{1}{2}\varphi, \qquad \varphi_2 = \varphi_{\mathrm{C}} - \frac{1}{2}\varphi, \tag{37}$$

and the Lagrangian becomes

$$L = \frac{mR^2}{2} \left(2\dot{\varphi}_{\mathrm{C}}^2 + \frac{1}{2}\dot{\varphi}^2 \right) + \frac{e^2 R}{c^2} \left(\dot{\varphi}_{\mathrm{C}}^2 - \frac{1}{4}\dot{\varphi}^2 \right) V_\varphi(\varphi). \tag{38}$$

We define $J_{\mathrm{C}} \equiv \partial L / \partial \dot{\varphi}_{\mathrm{C}}$ and $J \equiv \partial L / \partial \dot{\varphi}$ and get the (exact) Hamiltonian

$$H = J_{\mathrm{C}} \dot{\varphi}_{\mathrm{C}} + J \dot{\varphi} - L = \frac{1}{4} \frac{J_{\mathrm{C}}^2}{mR^2 \left(1 + \frac{e^2 V_\varphi(\varphi)}{mc^2 R} \right)} + \frac{J^2}{mR^2 \left(1 - \frac{e^2 V_\varphi(\varphi)}{mc^2 R} \right)}. \tag{39}$$

Clearly $\dot{J}_{\rm C} = -\partial H/\partial\dot{\varphi}_{\rm C} = 0$ so the center of mass (angular) momentum $J_{\rm C}$ is conserved. We put

$$|J_{\rm C}| \equiv 2J_{\rm F} = \text{const.} \tag{40}$$

and expand to first order in the parameter $\frac{e^2/R}{mc^2} = r_{\rm e}/R$. Throwing away a constant we end up with the following Hamiltonian for the relative motion of the electrons

$$H = \frac{J^2}{mR^2} - \frac{J_{\rm F}^2 - J^2}{mR^2}\frac{e^2}{mc^2}\frac{V_\varphi(\varphi)}{R}. \tag{41}$$

Consistency with our original assumptions requires that $J^2 \ll J_{\rm F}^2$ and thus we neglect the J^2 in the second term. Series expansion of V_φ gives

$$V_\varphi(\varphi) = \frac{1}{4}\frac{1+3\cos\varphi}{\sqrt{2(1-\cos\varphi)}} = \frac{1}{|\varphi|} - \frac{1}{3}|\varphi| + \frac{97}{5760}|\varphi|^3 + \dots, \tag{42}$$

for the angular potential energy, so near $\varphi = 0$ this is essentially a (one-dimensional) Coulomb potential. We keep the first term and introduce

$$p \equiv J/R, \;\; \mu \equiv m/2, \;\; r \equiv R\varphi, \;\; Z_{\rm F} \equiv \frac{J_{\rm F}^2/(mR^2)}{mc^2} = \frac{\mathcal{E}_{\rm F}}{mc^2}, \tag{43}$$

where $\mathcal{E}_{\rm F}$ is the Fermi energy. The Hamiltonian for the relative motion then becomes the well known Hamiltonian,

$$H = \frac{p^2}{2\mu} - \frac{Z_{\rm F}e^2}{|r|}, \tag{44}$$

for a (one dimensional) one electron atom with reduced mass μ and nuclear charge $Z_{\rm F}$.

The analysis above for two electrons on a circle can be done in an almost identical way in three dimensions [9, 10, 12] and shows that the Breit interaction can bind two electrons in their relative motion while their center of mass moves through the metal at the Fermi speed. The ground state energy in that case corresponds to a temperature of $\sim$ 0.1 mK. In the present one-dimensional case all parameters are the same except the dimensionality of the space. The one-dimensional hydrogen atom is treated in the literature [39, 40] and the ground state energy is known to go logarithmically to minus infinity when the dimension approaches one. To get a finite result we must therefore take account of the thickness, a, of our ring and change the potential to

$$V_1(r) = -\frac{Z_{\rm F}e^2}{|r|+a}. \tag{45}$$

In the three dimensional case the Bohr radius of the Hamiltonian (44) is $a_m = 2/Z_{\rm F} \approx 1.52\cdot 10^4\, r_s^2/a_0$, where a_0 is the ordinary Bohr radius and r_s the radius parameter. The three-dimensional ground state energy is $E_{3d} = -1/a_m^2 =$

$-Z_{\mathrm{F}}^2/4$. The corresponding result for the one-dimensional potential (45) is [39, 40]

$$E_{1d} = -\frac{1}{a_m^2}[2\ln(a_m/a)]^2. \tag{46}$$

The condition for this is that $a \ll a_m$. For the gold ring of Chandrasekhar et al. [20] with $a \approx 80\,\mathrm{nm}$ one finds that $a_m/a \approx 10^2$ using standard values for the Fermi energy of Au. One gets similar values for the Cu rings of Levy et al. [19]. The $1d$-condition is thus clearly satisfied in both experiments. We thus get that the ground state energy of the Darwin-Breit bound electron pairs corresponds to a temperature of roughly $1-2$ mK. This is a bit below the temperatures (7 mK) at which the persistent current gold ring experiments in [20] were performed, but the order of magnitude agreement is noteworthy. In the 10^7 Cu-rings experiment of Levy et al. [19] the temperature range $7-400$ mK was used. Physicists working with the theory of these phenomena can certainly not ignore the Darwin-Breit interaction and the possibility of pairing.

4.3 When does the Darwin-Breit term become large?

In the previous subsection we saw that the Darwin-Breit interaction, though small, can have important qualitative effect and lead to pairing of electrons. This effect is enhanced by one-dimensionality because of the logarithmic divergence of the $1/r$-interaction in one dimension. Let us now investigate the possibility of collective effects due to this term.

We return to the Lagrangian (32) and try to get the Hamiltonian without approximation. The generalized momentum is

$$J_i \equiv \frac{\partial L}{\partial \dot{\varphi}_i} = mR^2\dot{\varphi}_i + \frac{e^2 R}{c^2}\sum_{j(\neq i)}^{N} \dot{\varphi}_j V_{\varphi}(\varphi_i - \varphi_j). \tag{47}$$

In order to get an exact Hamiltonian we must solve for the $\dot{\varphi}_i$ in terms of the J_i. If we introduce the abbreviation $V_{ij} \equiv V_{\varphi}(\varphi_i - \varphi_j)$ we can write the N equations (47)

$$J_i = mR^2\left(\dot{\varphi}_i + \frac{r_{\mathrm{e}}}{R}\sum_{j(\neq i)}^{N} V_{ij}\dot{\varphi}_j\right), \qquad i = 1, \ldots, N, \tag{48}$$

(r_{e}=classical electron radius). As long as the sum here is negligible we have $J_i \approx mR^2\dot{\varphi}_i$ and easily find an approximate Hamiltonian. For few particles, small N, the sum will, in practice, never exceed the small number Nr_{e}/R by much, since in quantum mechanics the uncertainty principle prevents the V_{ij} from becoming to large. If, however, N is very large, the sum can still be small if the velocities $\dot{\varphi}_j$ have random signs.

We see that the condition for breakdown of the approximation $J_i \approx mR^2\dot{\varphi}_i$, and thus for important collective effects of the Darwin-Breit term, is that $Nr_{\rm e}/R$ no longer is small. A three dimensional estimate in [11] shows that, in fact, magnetic energy is minimized when

$$\frac{Nr_{\rm e}}{R} \sim 1 \tag{49}$$

where N is the number of correlated velocities. If we put $\varepsilon_e \equiv r_{\rm e}/R$ we can write equation (48) in the matrix form

$$\begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_N \end{pmatrix} = mR^2 \begin{pmatrix} 1 & \varepsilon_e V_{12} & \cdots & \varepsilon_e V_{1N} \\ \varepsilon_e V_{21} & 1 & \cdots & \varepsilon_e V_{2N} \\ \vdots & \vdots & & \vdots \\ \varepsilon_e V_{N1} & \varepsilon_e V_{N2} & \cdots & 1 \end{pmatrix} \begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \vdots \\ \dot{\varphi}_N \end{pmatrix}. \tag{50}$$

This shows that collective Darwin-Breit behavior is due to "off-diagonal long range order", a concept invented by C. N. Yang [41]. Here the concept reappears in a classical context and arises in the Legendre transformation from the Lagrangian, with a Darwin-Breit interaction, to the Hamiltonian.

In a real one-dimensional ring of atoms with electrons this cannot happen, as will be shown below. The algebra, however, is, barring notational and other irrelevant details, the same in two and three dimensions [10, 11]. We have already seen, in equation (30), that this parameter, $Nr_{\rm e}/R$, can be unity in three dimensions when the system is superconducting. Everything thus falls nicely into place. The Darwin-Breit term can lead to pairing of electrons at sufficiently low temperatures. Provided one has long range correlation of velocities it can also lead to a large collective effect, which, in fact, seems to be superconductivity.

The condition (49) will imply different physics for different spatial dimension d. The number N of ballistic, or semiclassical, or superconducting, or velocity-momentum correlated, electrons will be limited by the fact that there will be at most one contributed per atom, usually much less. Assume, for definiteness, the maximum number. For a sample of spatial dimension d and side length R this gives, very roughly,

$$N_{\rm max}(d) = R^d/a_0^d, \tag{51}$$

where a_0 is the Bohr-radius. If we put this in equation (49) we get $\frac{R^d r_{\rm e}}{a_0^d R} \sim 1$ which implies that

$$R^{d-1} \sim a_0^d/r_{\rm e}. \tag{52}$$

This gives the following (minimum) sizes R of superconducting structures in spatial dimension d

$$d \to 1+ \qquad \Rightarrow \qquad R \to \infty, \tag{53}$$

$$d = 2 \qquad \Rightarrow \qquad R \sim a_0^2/r_{\rm e} \approx 19000\, a_0 \approx 1\,\mu\text{m}, \tag{54}$$

$$d = 3 \qquad \Rightarrow \qquad R \sim a_0 \sqrt{a_0/r_{\rm e}} \approx 140\, a_0 \approx 10\, {\rm nm}. \tag{55}$$

As stated above, we see that $d = 1$ does not permit long range correlation. We saw that this does not mean that electron pairs do not form. It only means that no long range collective phenomenon (phase transition?) will be possible. Two dimensions differ from three in that structures (samples) must be at least two orders of magnitude larger in (linear) size.

5 Conclusions

The experienced theoretical physicist, should, just by looking at formula (1), see that there is trouble with the thermodynamics ahead, since the interaction is long range ($\sim 1/r$) and there is no natural screening mechanism similar to that which limits the range of the Coulomb interaction. This trouble is here identified with superconductivity. The main new point here, compared to the previous investigations by the author, is the discovery that the parameter $N r_{\rm e}/R$, of equation (49), which has appeared again and again in my study of the Darwin Hamiltonian (the exact Hamiltonian corresponding to the Lagrangian with the Darwin-Breit term), also miraculously appears in an estimate of the number of superconducting electrons, equation (30). This gives a direct connection to the heart of superconductivity that was missing before. The painful but only conclusion must be that the Darwin-Breit interaction is *the* interaction between electrons that causes superconductivity.

References

[1] M. Brack and R. K. Bhaduri, *Semiclassical Physics* (Addison Wesley, Reading, Massachusetts, 1997).

[2] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).

[3] C. G. Darwin, Phil. Mag. **39**, 537 (1920).

[4] G. Breit, Phys. Rev. **34**, 553 (1929).

[5] G. Breit, Phys. Rev. **39**, 616 (1932).

[6] J. Sucher, Advances in Quantum Chemistry **30**, 433 (1998).

[7] P. Strange, *Relativistic Quantum Mechanics, with applications in condensed matter and atomic physics* (Cambridge University Press, Cambridge, 1998).

[8] A. H. Cook, Proc. R. Soc. Lond. A **415**, 35 (1988).

[9] H. Essén, Physica Scripta **52**, 388 (1995).

[10] H. Essén, Phys. Rev. E **53**, 5228 (1996).

[11] H. Essén, Phys. Rev. E **56**, 5858 (1997).

[12] H. Essén, J. Phys. A: Math. Gen. **32**, 2297 (1999).

[13] H. Welker, Z. Physik **114**, 525 (1939).

[14] N. D. Mathur *et al.*, Nature **394**, 39 (1998).

[15] K. Capelle and E. K. U. Gross, Physics Letters A **198**, 261 (1995).

[16] M. Büttiker, Y. Imry, and R. Landauer, Phys. Lett. **96A**, 365 (1983).

[17] F. Bloch, Phys. Rev. B **2**, 109 (1970).

[18] R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, Phys. Rev. Lett. **54**, 2696 (1985).

[19] L. P. Levy, G. Dolan, J. Dunsmuir, and H. Bouchiat, Phys. Rev. Lett. **64**, 2074 (1990).

[20] V. Chandrasekhar *et al.*, Phys. Rev. Lett. **67**, 3578 (1991).

[21] B. L. Johnson and G. Kirczenow, Can. J. Phys. **76**, 173 (1998).

[22] Y. Imry and M. Peshkin, in *Electron, a centenary volume*, edited by M. Springford (Cambrige University Press, Cambridge, 1997), pp. 208–236.

[23] Y. Imry, *Introduction to mesoscopic physics* (Oxford, New York, 1997).

[24] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Pergamon, Oxford, 1976).

[25] M. R. Geller, Phys. Rev. B **53**, 9550 (1996).

[26] H. Essén, Physica Scripta **40**, 761 (1989).

[27] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).

[28] F. London, *Superfluids, Volume 1, Macroscopic Theory of Superconductivity*, 2nd ed. (Dover, New York, 1961).

[29] D. J. Thouless, *Topological quantum numbers in nonrelativistic physics* (World Scientific, Singapore, 1998).

[30] N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961).

[31] B. S. Deaver, Jr. and W. M. Fairbank, Phys. Rev. Lett. **7**, 43 (1961).

[32] R. Doll and M. Näbauer, Phys. Rev. Lett. **7**, 51 (1961).

[33] C. E. Gough *et al.*, Nature **326**, 855 (1987).

[34] R. M. Brady, Journal of Low Temperature Physics **49**, 1 (1982).

[35] B. Cabrera and M. E. Peskin, Phys. Rev. B **39**, 6425 (1989).

[36] M. Liu, Phys. Rev. Lett. **81**, 3223 (1998).

[37] J. Tate, B. Cabrera, S. Felch, and J. T. Anderson, Phys. Rev. Lett. **62**, 845 (1989).

[38] M. A. Sanzari, H. L. Cui, and F. Karwacki, Appl. Phys. Lett. **68**, 3802 (1996).

[39] L. K. Haines and D. H. Roberts, Am. J. Phys. **37**, 1145 (1969).

[40] A. R. P. Rau, Am. J. Phys. **53**, 1183 (1985).

[41] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).