

Homework 3, due 22/2-2008

We are going to solve the equations for an unsteady isentropic flow through a convergent-divergent nozzle using the MacCormack technique. This problem is described in detail in *Anderson* chapter 7.

The governing equations on conservative form are, (see *Anderson* equations (7.15), (7.93) and (7.96))

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{G}}{\partial x} = \mathbf{F}$$

where

$$\mathbf{U} = \begin{pmatrix} \rho A \\ \rho Av \\ \rho(e + v^2/2)A \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho Av \\ \rho Av^2 + pA \\ \rho(e + v^2/2)Av + pAv \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} 0 \\ p \frac{\partial A}{\partial x} \\ 0 \end{pmatrix}$$

$$A(x) = 1 + 2.2(x - 1.5)^2, \quad x \in [0, L]$$

$L = 3$ is the length of the nozzle. We also need relations from thermodynamics; $p = \rho RT$ and $e = RT/(\gamma - 1)$ where $R = 288.7 \text{ J/kg/K}$ is the gas constant and $\gamma = 1.4$.

Initial conditions:

$$v(0, x) = 0 \quad (1)$$

$$p(0, x) = p_0 + \frac{x}{L}(p_e - p_0) \quad (2)$$

$$T(0, x) = T_0 \quad (3)$$

where $p_0 = 101000 \text{ Pa}$ and $T_0 = 300 \text{ K}$ (the reservoir conditions). p_e is the exit (back) pressure. The size of the pressure ratio p_0/p_e determines the flow through the nozzle, so p_e has to be chosen differently depending on the flow case. To compute $\rho(0, x)$ and $e(0, x)$ the given initial conditions can be used together with the thermodynamical relations given above.

Boundary conditions:

To determine the number of boundary conditions required at each boundary we use the method of characteristics.

The inflow will always be subsonic, i.e. there are two in-going characteristics and one outgoing. Therefore, we specify the value of two of the variables, e.g.

$$\rho(t, 0) = \rho_0 = \frac{p_0}{RT_0} \quad (4)$$

$$T(t, 0) = T_0 \Rightarrow e(t, 0) = \frac{RT_0}{\gamma - 1} \quad (5)$$

For the third variable, v , we need a numerical boundary condition which is obtained by linear extrapolation.

The outflow conditions will depend on the flow case;

For **supersonic** outflow all characteristics are going out from the domain and no physical boundary conditions are required. Values of the flow variables at the boundary are obtained by linear extrapolation.

For **subsonic** outflow we have one in-going characteristic and we prescribe the pressure to be equal to the exit pressure

$$p(t, L) = p_e. \quad (6)$$

The other two boundary conditions are obtained by using conditions that are compatible with the physical behavior of the flow. For inviscid flow,

$$\frac{p}{\rho^\gamma} = s$$

is constant along a stream line. We also know that along one of the outgoing characteristics the quantity

$$R2 = u + \frac{2c}{\gamma - 1}$$

is transported at velocity $u + c$ where $R2$ is called a Riemann invariant. These facts are used to construct numerical boundary conditions. We extrapolate the values of the entropy and the Riemann invariant at the boundary by using their values inside the domain as

$$s_m = 2s_{m-1} - s_{m-2} \quad (7)$$

$$R2_m = 2R2_{m-1} - R2_{m-2} \quad (8)$$

where m is the number of grid points. From these relations we can compute ρ , ρv and $\rho(e + v^2/2)$ at the boundary.

$$\begin{aligned} \rho_m &= \left(\frac{p_e}{s_m} \right)^{\frac{1}{\gamma}} \\ v_m &= R2_m - \frac{2\sqrt{\frac{\gamma p_e}{\rho_m}}}{\gamma - 1} \\ e_m &= \frac{p_e}{\rho_m(\gamma - 1)} \end{aligned}$$

It is your job to write a code and to run the three different cases:

1. Supersonic outflow, $p_e = 0.1278p_0$
2. Subsonic outflow without shock, $p_e = 0.98p_0$
3. Subsonic outflow with a shock, $p_e = 0.6784p_0$

The MacCormack scheme should be used and the code can either be based on the code handed out for the *Shock tube* or you can write a code completely on your own.

For each case you should run the code until you obtain a **steady solution** and present the results, see e.g. Figure 1 (b). Be sure to run enough time steps to reach convergence to steady state (around 5000 to 10000). Check the density residual, $\|\rho^n - \rho^{n-1}\|/\|\rho^n\|$, where $\|\rho^n\|$ is the norm of the vector containing ρ_j^n , $j = 1, 2, \dots, m$, (use **norm** in MATLAB), and make sure it has decreased at least five orders of magnitude, see Figure 1 (a).

We know from *Homework 2* that artificial viscosity is needed to obtain a solution free from oscillations and that it should be implemented using a conservative formulation,

$$\mathbf{U}_t + \mathbf{G}(\mathbf{U})_x + (\mathbf{KU}_x)_x = \mathbf{F}$$

Use the same artificial viscosity model as in the *Homework 2* but use a pressure switch instead of the density.

Experiment with the amount of artificial viscosity to obtain the "best" result.

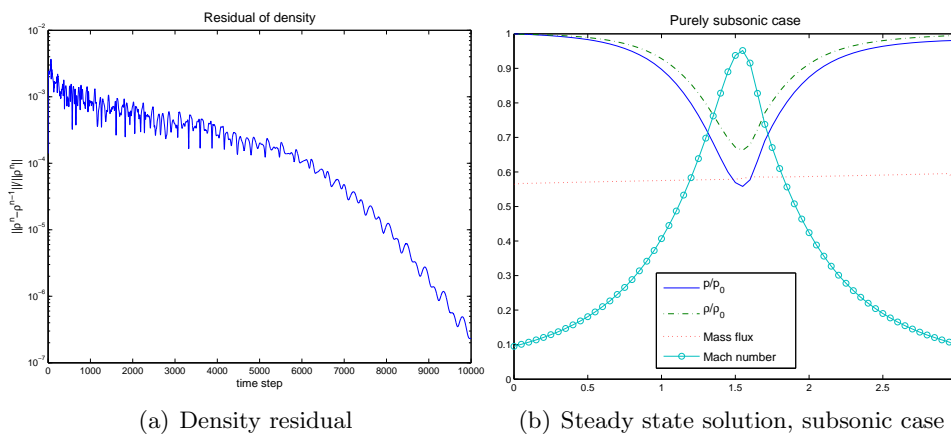


Figure 1: Subsonic case