Homework 1B, due 11/2-2008

1.) Solve the linear wave equation

$$\begin{aligned} u_t + au_x &= 0 \quad a = 1 \quad x \in (0,1), \quad t > 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

with periodic boundary conditions, $u(0,t) = u(1,t)$, numerically by the upwind scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

and initial data

$$u(0,x) = \begin{cases} 1 & \text{if} \quad 0.25 \le x \le 0.5 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Use 100 grid points, 20 time steps and a Courant number, $CN = a\Delta t/\Delta x = 0.4, 0.8$ and 1.0. Hand in plots of the numerical solution togehter with the exact solution which demonstrate how the dissipation varies with the Courant number. Explain the behavior of the numerical solution by the concept of modified equations.

2.) Solve equation (1) with initial data given by

$$u(0,x) = e^{-(x-0.3)^2/0.01} \tag{3}$$

and use periodic boundary conditions. Again, use the upwind scheme. Advect the pulse on 100 grid points and 20 time steps and verify that $CN \le 1$ is the stability limit for the upwind scheme. Hand in plots of the numerical solution together with the exact solution. Run the problem with $CN = 1.6$. What does the numerical solution look like ? Why ?

3.) Solve equation (1) with initial data (2) using the Lax-Wendroff scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n) + \frac{(a\Delta t)^2}{2\Delta x^2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

What is the stability limit for this scheme? Verify numerically.

Now run the problem with $CN = 0.4, 0.8$ and 1.0. Note the waves behind the step in your numerical solution. These are related to numerical dispersion.

Dispersion is related to the wave speeds of the solution. When the Lax-Wendroff scheme is used to approximate the linear wave equation, the wave speed is changed from

$$a \quad \text{to} \quad a + \frac{1}{6}\Delta x^2 a(CN^2 - 1)\kappa^2 \tag{4}$$

where κ is the wave number.

Show (4) by looking at the modified equation and explain why the spurious waves in the solution of (2) appear *behind* the step.