Homework 4

Projection method and staggered grid for incompresible flows

due 27/2-2008

When solving the incompressible Navier-Stokes equations, other techniques are normally used compared to those used for the solution of the compressible Navier-Stokes equations. One common method, used to ensure that the velocity field is divergence free, is the projection (or pressure correction) method.

Task 1:

Your first task will be to derive the pressure correction formula.

Consider the incompressible Navier-Stokes equations in two dimensions

$$u_x + v_y = 0 \quad (1)$$
$$u_t + (u^2)_x + (uv)_y + p_x = \frac{1}{Re}\Delta u + f \quad (2)$$
$$v_t + (uv)_x + (v^2)_y + p_y = \frac{1}{Re}\Delta v + g \quad (3)$$

where u and v are the velocity components in the x, and y-direction, p is the pressure, Re is the Reynolds number and f and g are given source functions. Assume that the equations are solved with some given boundary and initial conditions. Suppose that the equations are discretized in space by the finite volume approximation. The discretized equations are solved in time by the Marker and Cell (MAC) method:

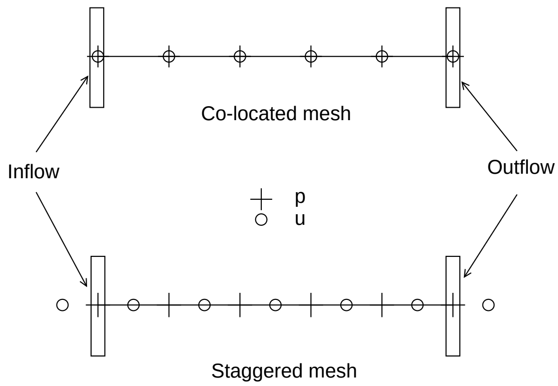
$$u^{n+1} = \underbrace{u^n - dt[(u^n)^2_x + (u^n v^n)_y] + \frac{\Delta t}{Re}\Delta u^n + f^n}_{F^n} - dt p^{n+1}_x \quad (4)$$
$$v^{n+1} = \underbrace{v^n - dt[(u^n v^n)_x + (v^n)^2_y] + \frac{\Delta t}{Re}\Delta v^n + g^n}_{G^n} - dt p^{n+1}_y$$

yielding

$$u^{n+1} = F^n - dt p^{n+1}_x, \quad (5)$$
$$v^{n+1} = G^n - dt p^{n+1}_y.$$

By substituting the velocities at t^{n+1} into the continuity equation (1), show that the pressure at this time is given by the Poisson equation

$$\Delta p^{n+1} = \frac{1}{dt}(F^n_x + G^n_y) \quad (6)$$



Task 2:

You will in this homework learn the importance of using a staggered grid when solving an incompressible flow problem (*see Anderson, p. 250-253*).

A strongly simplified one-dimensional analogue to the incompressible Navier-Stokes equations (neglecting the viscous and non-linear convection terms) is given by

$$u_t = -p_x, \tag{7}$$

$$u_x = 0, \tag{8}$$

with the initial and boundary conditions

$$u(t=0,x) = 1, \qquad p(t,x=0) = 1.$$

The spatial domain is $x \in [0,1]$. Your task is

1. to derive the expression for projection methods for equations (7) and (8),
2. to write a small program that solves these equations (derived above in 1).using a FD central discretization in space and explicit Euler in time on
   (a) a standard grid
   (b) a staggered grid

The idea is to demonstrate the existence of spurios checkboard solutions. Sometimes, if the initial solution is exact you may not observe these solutions. Therefore, perturb your initial condition with some noise of low amplitude ($\sim \mathcal{O}(0.01)$).

Write down the discretized equations at boundaries so one can see how the boundary conditions have been implemented.

Plot pressure and velocity field at different time steps. Compare the result from (a) and (b) and explain the different behavior of the solutions.

Hints:

- Solve for the interior points and use the pressure boundary conditions at $x = 0$ to compute p_{-1} and set $p_{N+1} = p_N$. The latter results to a Neumann condition for pressure at $x = 1$. Value of u_{N+1} can be approximated by extarpolation, if necessary.
- The MATLAB function `rand` produces random noise.
- Use different $\frac{\Delta t}{\Delta x}$, any effects?.
- Integrate the equations for a sufficient long time such that you can see the spurios solution grow or disappear totally.