Study questions

1. Define and use the Rankine-Hugoniot jump condition to compute the shock speed for the following problem

$$
\begin{aligned}
u_t + uu_x &= 0 \quad -\infty < x < \infty, \quad t > 0 \\
u(x,0) &= \begin{cases} 1 & x \leq 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

2. Define the entropy condition for a scalar conservation law.

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

with a convex flux function $f(u)$. The shock is moving with speed s and the state to the left is given by u_L and the state to the right by u_R.

Why do we need an entropy condition ?

3. Define a total variation decreasing (TVD) method. Why is this a desirable property ?

4. Investigate the one-sided difference scheme

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

for the advection equation

$$u_t + au_x = 0$$

Consider the cases $a > 0$ and $a < 0$.

a) Prove that the scheme is consistent and find the order of accuracy. Assume $\Delta t/\Delta x$ constant.

b) Determine the stability requirement for $a > 0$ and show that it is unstable for $a < 0$.

5. Apply Lax-Friedrichs scheme to the linear wave equation

$$u_t + au_x = 0$$

that is,

$$u_j^{n+1} = \frac{1}{2}(u_{j-1}^n + u_{j+1}^n) - \frac{a\Delta t}{2\Delta x}(u_{j+1}^n - u_{j-1}^n)$$

a) Write down the modified equation.

b) What type of equations is this?

c) What kind of behavior can we expect from the solution?

6. A the three-point centered scheme applied to

$$u_t + au_x = 0, \quad a > 0.$$

yields the approximation

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{2\Delta x}(u_{j+1} - u_{j-1})$$

Show that this approximation is not stable even though the CFL condition is fulfilled.

7. What does Lax's equivalence theorem state?

8. What is the condition on the $n \times n$ real matrix $A(\mathbf{u})$ for the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

to be hyperbolic ?

9. The barotropic gas dynamic equations

$$\begin{aligned} \rho_t + \rho u_x &= 0 \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0 \end{aligned} \tag{1}$$

where

$$p = p(\rho) = C\rho^\gamma$$

i and C a constant, can be linearized by considering small perturbations (ρ', u') around a motionless gas.

a) Let $\rho = \rho_0 + \rho'$ and $u = u_0 + u'$ where $u_0 = 0$. Linearize the system (1) and show that this yields the following linear system (the primes has been dropped)

$$\begin{aligned} \rho_t + \rho_0 u_x &= 0 \\ u_t + \frac{a^2}{\rho_0}\rho_x &= 0 \end{aligned} \tag{2}$$

where a is the speed of sound. a and ρ_0 are constants.

b) Is the system given by (2) a hyperbolic system? Motivate your answer.

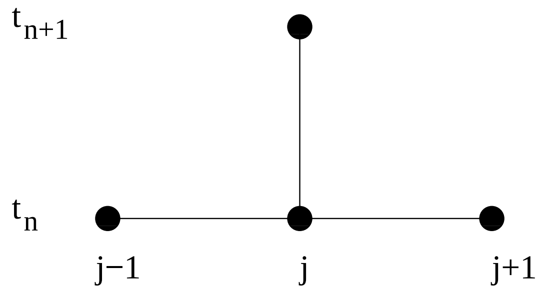
c) Determine the characteristic variables in terms of ρ and u.

d) Determine the partial differential equations the characteristic variables fulfill - characteristic formulation.

e) Given initial conditions at $t = 0$ and let $-\infty < x < \infty$ (no boundaries)

$$\rho(0, x) = sin(x) \quad u(0, x) = 0$$

determine the analytical solution to (2) for $t > 0$. Hint: Start from the characteristic formulation.



10. The linearized form of (2) is given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \qquad (3)$$

where a is the speed of sound. a and ρ_0 are constants.

a) Draw the domain of dependence of the solution to the system (3) in a point P in the x-t plane.

b) The system is solved numerically on a grid given by $x_j = j\Delta x, j = 0, 1, 2...$ and $t_n = n\Delta t, n = 0, 1, 2, ...$ using an explicit three-point scheme, see the figure below.

Draw the domain of dependence of the numerical solution at P (in the same figure as a)) of the three-point scheme in the case when

i) the CFL condition is fulfilled

ii) the CFL condition is NOT fulfilled.

Assume that P is a grid point.

11. To solve Euler equations in 1D

$$\rho_t + \rho u_x + u\rho_x = 0$$
$$u_t + uu_x + \frac{1}{\rho}p_x = 0$$
$$p_t + \rho c^2 u_x + up_x = 0$$

How many boundary conditions must be added at (motivate your answer)

inflow boundary when the flow is

a) Supersonic

b) Subsonic

outflow boundary when the flow is

c) Supersonic

d) Subsonic

12. Projection on a divergence-free space

a) Show that a vector field w_i can be decomposed into

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

where u is is divergence free and parallel to the boundary.

b) Apply this to the Navier-Stokes equations, show that the pressure term disappears and recover an equation for the pressure from the gradient part.

13. From the differential form of the Navier-Stokes equations obtain the Navier-Stokes equations in integral form used in finite-volume discretizations,

14. Finite volume (FV) discretization

    (a) Derive the finite volume (FV) discretization on arbitrary grids of the continuity equation ($\partial u_i / \partial x_i = 0$),

    (b) derive the FV discretization for Laplace equation on a Cartesian grid,

    (c) show that both are equivalent to a central difference approximation for Cartesian grids.

15. State the difficulties associated with the the finite-volume discretizations of the Navier-Stokes equations on a colocated grid? and show the form of the spurious solution which exist.

16. Staggered grid

    (a) Define an appropriate staggered grid that can be used for the discretization of the Navier-Stokes equations,

    (b) write down the FV discretization of the Navier-Stokes equations on a staggered cartesian grid,

    (c) discuss how to treat noslip and inflow/outflow boundary conditions.

17. Time dependent flows.

    (a) Define a simple projection method for the time dependent Navier-Stokes equations

    $$\frac{d}{dt}\begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{N}(u) & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

    (b) show in detail the equation for the pressure to be solved at each time step and discuss the boundary conditions for the pressure.

18. Time step restriction for Navier-Stokes solutions.

    (a) Motivate the use of an appropriate form of the advection-diffusion equation as a model equation for stability analysis,

    (b) derive the time step restrictions for the 1D version of that equation,

    (c) state the 2D equivalent of that restriction.

19. Iterative techniques for linear systems.

    (a) Define a distributive iteration method for the linear system $Ay = b$.

    (b) Define Gauss-Seidel iterations for the Laplace equation, discuss the convergence rate and derive an approximation for number of iterations required for error reduction of $\mathcal{O}(h^2)$.

    (c) Define the 2-level multigrid method for the Laplace equation,

20. Coordinate transformation

    (a) Define the coordinate transformation from a Cartesian one (x, y, z) to a general one (ξ, η, ζ). State the Jacobian matrix of transformation and describe a practical way of computing it.

    (b) Derive the transformation of the 2D Navier-Stokes equations

$$\text{from} \quad \frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = 0 \quad \text{to} \quad \frac{\partial \mathbf{U}'}{\partial t} + \frac{\partial \mathbf{F}'}{\partial \xi} + \frac{\partial \mathbf{G}}{\partial \eta} = 0,$$

    and give the vectors $\mathbf{U}', \mathbf{F}'$ and $\mathbf{G}'$ in terms of $\mathbf{U}, \mathbf{F}$ and $\mathbf{G}$.

21. Compact finite-difference scheme

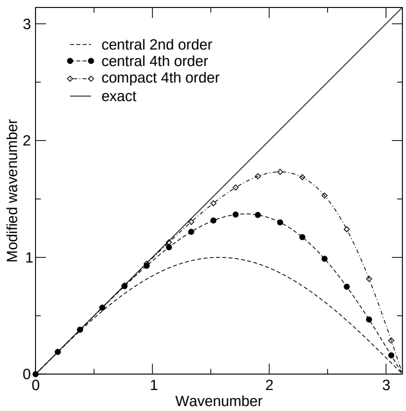
    Consider the general approximation of type

$$\beta(f'_{i+2} + f'_{i-2}) + \alpha(f'_{i+1} + f'_{i-1}) + f'_i =$$
$$\frac{c}{6h}(f_{i+3} - f_{i-3}) + \frac{b}{4h}(f_{i+2} - f_{i-2}) + \frac{a}{2h}(f_{i+1} - f_{i-1}),$$

    (a) and derive the equations which should be satisfied to get different order of accuracy for discretization of first derivative f'_i.

    (b) By Fourier analysis of the differencing error of the scheme above derive a expression for the modified wavenumber.

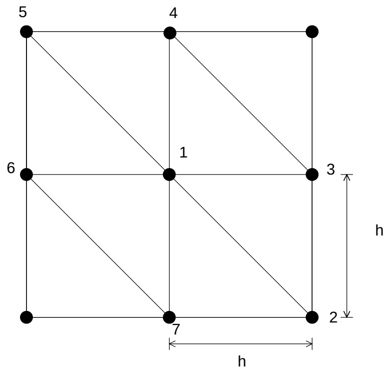
    (c) What do the curves in the figure tell us?



22. Unstructured Node-Centered finite volume.

    (a) Define the dual grid.

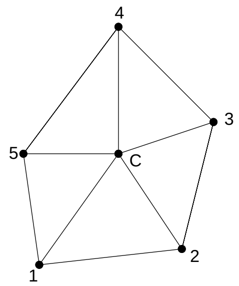
(b) Present a finite-volume approximation of $u_t = u_{xx} + u_{yy}$. Examine the consistency of the scheme and give the order of the accuracy (use the grid given here).



(c) Show that the u_x at node c can be approximated by the following finite-volume approximation and proof that its accuracy is $\mathcal{O}(h)$ (first order),

$$(u_x)_c \approx \frac{1}{V_c} \sum_i \frac{u_c + u_i}{2} \delta y_i.$$

(V_c is the volume of the dual grid)



23. Upwind discretization

(a) Consider equation $u_t + au_x = 0$, where a is the convective velocity. Give a first-order accurate upwind discretization of his equation which is stable independent of the sign of a.

(b) Define a flux spliting scheme for discretization of one-dimensional Euler equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} = 0, \quad \mathbf{U} = \begin{pmatrix} \rho \\ \rho u \\ E_t \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} \rho u \\ \rho u^2 \\ (E_t + p)u \end{pmatrix}.$$