Control and estimation using stochastic methods

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Flow systems of engineering interest are often exposed to disturbances that are erratic, unpredictable, and thus conveniently described by their statistics. When aiming at controlling these flow systems, as for example to prevent transition to turbulence, it is natural to adopt a framework in which one seek to affect the statistics of the flow itself. In this talk we describe an optimal feedback control framework based on stochastic optimization.

Modeling: A feedback controller uses instantaneous information about the flow field to decide the best action. The information is measured using sensors, as for instance wall skin friction measurement, and act on the flow using actuators, as for instance blowing and suction at the wall. The feedback law is the relation between sensors and actuator. In this talk we will consider linear controllers, for which we assume that the dynamics of small fluctuations about a base flow profile is linear.

Stochastic disturbances: The external disturbances can be described by their statistics, assuming that they are Gaussian distributed with zero mean, we represent them by their covariance in the flow domain. The covariance matrix of a spatial random process has on its diagonal the variance of the process along the spatial directions, and the off-diagonal terms represent the covariance of the process at two spatial locations (two-point correlation).

The state (velocity, pressure) of a linear system excited by a random process is itself a random process that can be described by its statistics. Consider for instance the dynamic system with state q, and stochastic input w

$$\dot{q} = Aq + B_1 w,$$

where A is the dynamic operator, and B_1 represent how the disturbances enter the system. The Lyapunov equation describes the covariance of the state of a linear system excited by random inputs

$$AP + PA^H + B_1 W B_1^H = 0,$$

where W is the covariance of the external disturbances w, and P is the covariance of the state. For instance, A could denote the Orr-Sommerfeld/Squire operator, with the state being $(\hat{v}, \hat{\eta})$ the Fourier transforms of the wall normal velocity/wall normal vorticity in the context of the Poiseuille flow.

Optimisation of the statistics: Introducing actuators and sensors in the flow system, we build the two dynamic systems

$$\text{flow: } \begin{cases} \dot{q} = Aq + B_1 w + B_2 u \\ y = Cq + g \end{cases}, \quad \text{estimator: } \begin{cases} \dot{\hat{q}} = A\hat{q} - v \\ \hat{y} = C\hat{q} \end{cases} \tag{1}$$

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where u and y are the actuator and sensor signals and v is the estimator forcing. The stochastic inputs are the external disturbances w with covariance W, and the sensor noise g with covariance G. For full information control, we assume that the flow state q is known. In a real situation, this information is not available, so we need to estimate the flow state using an estimator, with state $\hat{q}$.

From this representation, we define two *design* problems: find a state feedback control gain $u = Kq$, and an estimation gain $v = L(y - \hat{y})$ such that the controlled flow and the estimation error $\tilde{q} = q - \hat{q}$ have low kinetic energy. Introducing K and L in (1) we obtain the dynamics for the controlled flow and for the estimation error

$$\dot{q} = (A + B_2K)q + B_1w, \quad \dot{\tilde{q}} = (A + LC)\tilde{q} + B_1w - Lg. \tag{2}$$

From these equations, we see that we should find K such that $A + B_2K$ is stable and that the resulting dynamics is not sensitive to the stochastic disturbances B_1w, and that we should find L such that $A + LC$ is stable and is not sensitive to the stochastic disturbances $B_1w - Lg$.

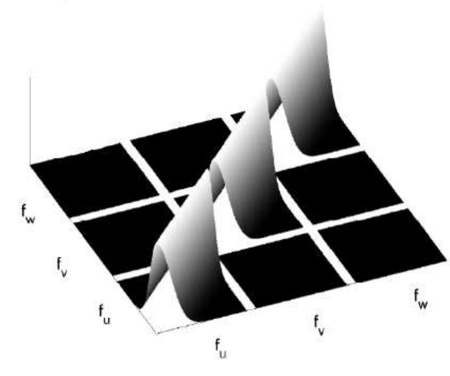
From (2) we obtain two Lyapunov equations for the covariance P of the controlled flow state and the covariance $\tilde{P}$ of the estimation error

$$\begin{aligned} \text{Control: } & (A + B_2K)P + P(A + B_2K)^H + B_1WB_1^H = 0, \\ \text{Estimation: } & (A + LC)\tilde{P} + \tilde{P}(A + LC)^H + B_1WB_1^H - LGL^H = 0. \end{aligned}$$

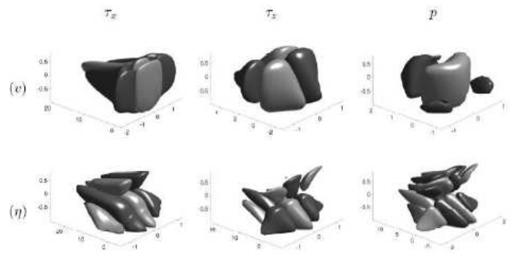
Defining two objective functions based on P and $\tilde{P}$, we can express the optimization as the minimisation of a Lagrangian built from these two Lyapunov equations. The optimal gains are finally obtained as solutions of two Riccati equations.

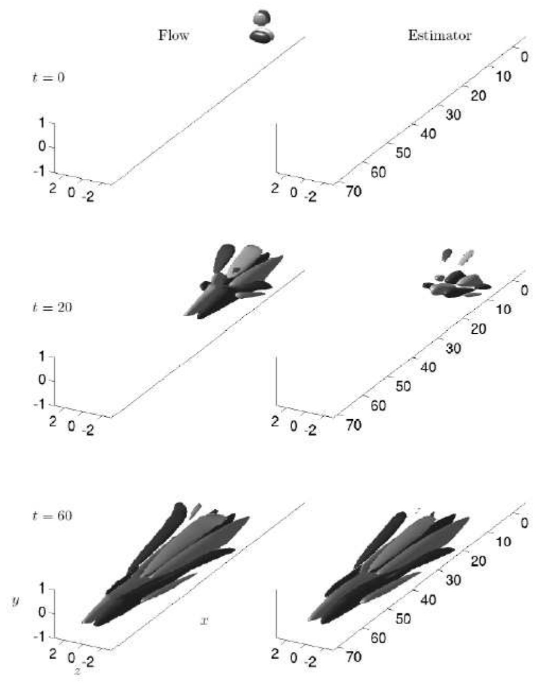
Example: As an illustrative example, we perform estimation of a localised initial condition in Poiseuille flow. See below for a simple covariance model of a disturbance forcing on the $u, v, w)$ velocity components, for the estimation convolution kernels obtained after backward Fourier transform of L to physical space. And for the time evolution of the estimation of a localised initial condition.

Simple covariance model:



Estimation convolution kernels:





Estimation of initial condition