Exam, SG2214 Fluid Mechanics
8 January 2013, at 9:00-13:00
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Copies of Cylindrical and spherical coordinates*, which will be supplied if necessary, can be used for the exam as well as a book of basic math formulas, a calculator and an English dictionary. The point value of each question is given in paranthesis and you need more than 20 points to pass the exam including the points obtained from the homework problems.*

1. (10 p.) The velocity field in a fluid is given by $\boldsymbol{u} = \alpha x^2 \boldsymbol{e_x} - 2\alpha xy \boldsymbol{e_y}$ in a Cartesian coordinate system, where α is a positive constant.
a) Determine the rate of strain tensor and the rotation tensor.
b) Find the equation for the streamlines and make a sketch of them in a Cartesian coordinate system showing the direction of the flow.
c) Calculate the circulation, $\Gamma = \oint_S \boldsymbol{u} \cdot d\boldsymbol{x}$, along a closed circle S of radius r around the origin.

2. (10 p.) The velocity field in a fluid is given by $\boldsymbol{u} = \alpha(t)x^2\boldsymbol{e_x} - 2\alpha(t)xy\boldsymbol{e_y}$ in a Cartesian coordinate system, where $\alpha(t)$ is a time dependent function.
a) Give an expression for the rate of change with time of the velocity vector measured at the fixed point $\boldsymbol{x} = (L, L)$?
b) Give an expression for the rate of change with time of the velocity vector of a fluid element just as it passes the fixed point $\boldsymbol{x} = (L, L)$?
c) Give an expression for the rate of change with time of the velocity vector measured by an observer moving at constant speed $\boldsymbol{u}_{\text{observer}} = U_{\text{obs}}\boldsymbol{e_x}$ as the observer passes the fixed point $\boldsymbol{x} = (L, L)$?

3. (10 p.) Consider the flow of an incompressible, viscous fluid through two circular tubes according to the figure. The flow is forced in the larger tube by a cylinder at velocity U_0 and exits at B into the atmosphere through the narrow tube.

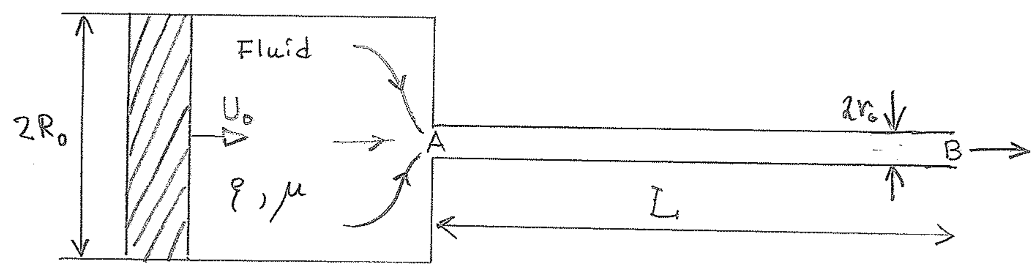


FIGURE 1

a) What is the flow rate $[m^3/s]$ at B?
b) Use the Navier-Stokes equations to derive an expression for the pressure difference between A and B assuming a fully developed viscous flow in the narrow tube?
c) What is the force on the narrow tube AB from the fluid inside it?

4. (10 p.) Consider the uniform flow $\boldsymbol{u}_\infty = U_\infty \boldsymbol{e_x}$ at large Reynolds number past a 2D cylindrical body which cross section is that of half a circle of radius a with the circular surface facing the flow and the flat surface facing downstream. Suppose the pressure on the circular part can be approximated to that of ideal irrotational flow around a full circular cylinder, and that the pressure on the flat part is constant given by the pressure at the points of separation of the boundary layers on the upper- and lowermost points of the cylindrical body. This pressure may also be taken approximately from the irrotational flow solution at the upper- and lowermost points of the full circular cylinder. The pressure coefficient for irrotational flow around a circular cylinder without circulation is given by $c_p = \frac{p-p_\infty}{\frac{1}{2}\rho U_\infty^2} = 1 - 4sin^2\theta$, where p is the pressure at the cylinder surface and p_∞ is the free stream pressure.

a) Calculate the drag D' per unit width of the body.

b) Generally for a boundary layer, under what conditions do you expect separation of the boundary layer. Give a criteria which defines the point of separation of a boundary layer.

5. (10 p.) The asymptotic suction boundary layer in a incompressible fluid is given by

$$u = U_\infty \left(1 - e^{-\eta}\right);\ \eta = \frac{y}{\nu/V_\infty}$$
$$v = -V_\infty.$$

It represents the viscous flow above an infinite porous plane wall at $y = 0$ where a suction velocity $-V_\infty \boldsymbol{e_y}$ is applied perpendicular to the wall. The tangential velocity at the wall satisfies the no slip condition. Far away from the wall the velocity is constant $\boldsymbol{u}_\infty = U_\infty \boldsymbol{e_x} - V_\infty \boldsymbol{e_y}$. The temperature of the incoming fluid is T_∞ and the temperature of the porous wall is $T_{\text{wall}} = T_\infty + \Delta T$, which is also carried over to the fluid just passing into the pores of the wall. The fluid temperature is then

$$T = T_\infty + \Delta T e^{-\eta Pr} + \frac{U_\infty^2}{2c_{\text{p}}} \frac{Pr}{Pr-2}\left(e^{-2\eta} - e^{-\eta Pr}\right)$$

where the Prandtl number is defined $Pr = \nu/\kappa = \mu c_{\text{p}}/k$ ($\neq 2$ in this case). In an incompressible fluid $c_{\text{p}} = c_{\text{v}}$ and $e = c_{\text{p}}T$. Consider a cubical control volume L^3 where L is much larger than the boundary layer thickness $\delta = \max\{\nu/V_\infty, \nu/(V_\infty Pr)\}$ with one face adjacent to the porous wall. Evaluate each of the integrals in the energy equation for the control volume below and show that the given solution satisfies this equation.

$$\frac{d}{dt}\int_V \rho(e+\frac{1}{2}u_iu_i)dV + \oint_S \rho(e+\frac{1}{2}u_iu_i)(u_kn_k)dS = \int_V \rho g_iu_idV + \oint_S u_i\left(-p\delta_{ik} + \tau_{ik}\right)n_kdS - \oint_S q_in_idS.$$

$$\tau_{ik} = 2\mu(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll}) + \mu_B\delta_{ik}e_{ll}, \quad \text{and} \quad q_i = -k\frac{\partial T}{\partial x_i},\ e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

You may disregard effects of gravity.

Solutions Exam SG2214 130108

$\boxed{1.}$ $\underset{\sim}{u} = \alpha x^2 \underset{\sim}{e_x} - 2\alpha xy \underset{\sim}{e_y}$

a) $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ rate of strain tensor

$e_{11} = \frac{\partial u}{\partial x} = \alpha 2x$ $\quad e_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) =$

$= \frac{1}{2}(0 - 2\alpha y) = -\alpha y$

$e_{21} = e_{12} = -\alpha y$ $\quad e_{22} = \frac{\partial v}{\partial y} = -2\alpha x$

$$e_{ij} = \begin{bmatrix} \alpha 2x & -\alpha y \\ -\alpha y & -\alpha 2x \end{bmatrix}$$

$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ rotation tensor

$\xi_{11} = 0$ $\quad \xi_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = \alpha y$

$\xi_{21} = -\xi_{12} = -\alpha y$ $\quad \xi_{22} = 0$

$$\xi_{ij} = \begin{bmatrix} 0 & \alpha y \\ -\alpha y & 0 \end{bmatrix}$$

1 b)

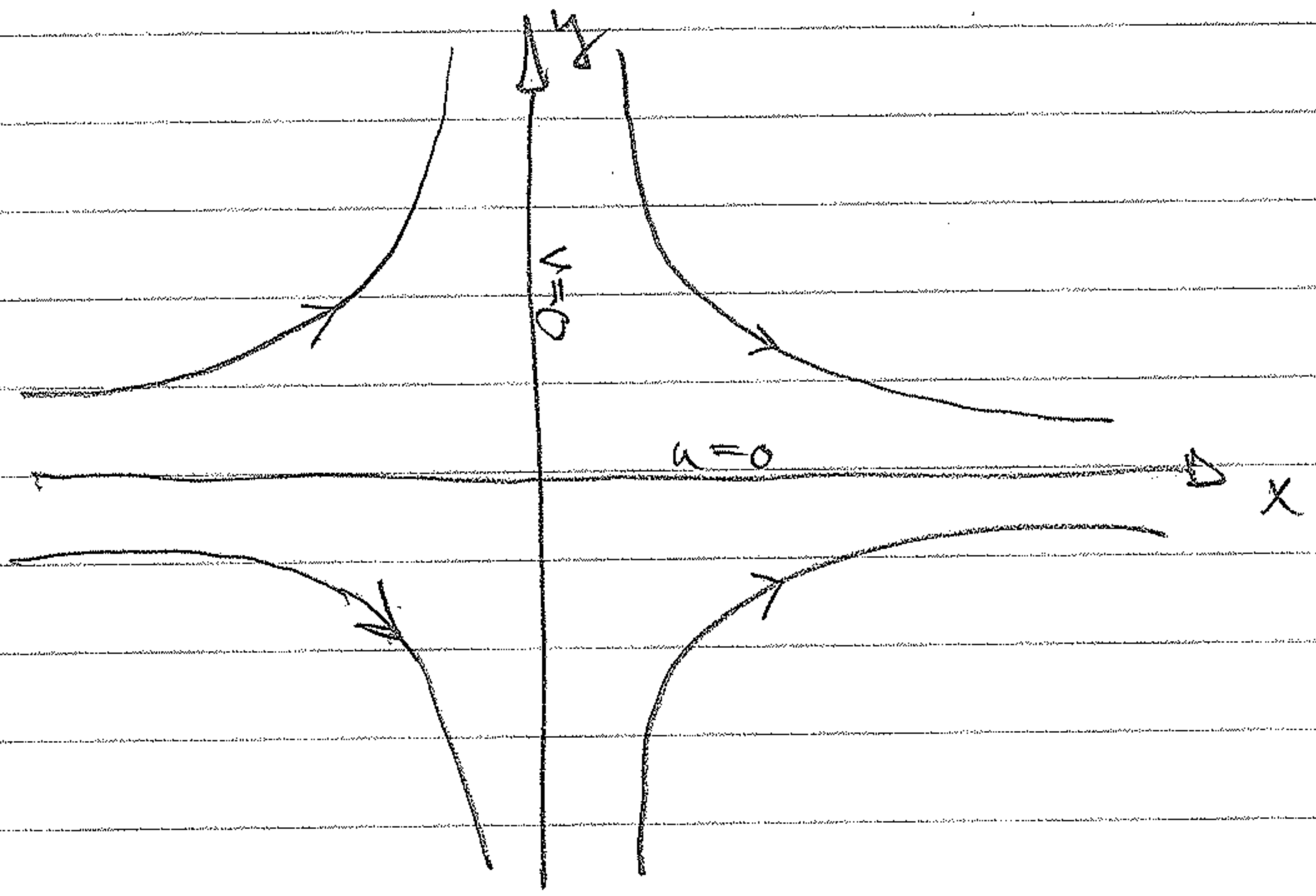
Streamlines $\frac{dx}{u} = \frac{dy}{v}$

$$\Rightarrow \frac{dx}{\alpha x^2} = -\frac{dy}{2\alpha x y}$$

$$\Rightarrow \frac{dx}{x} = -\frac{dy}{2y}, \Rightarrow \ln x = -\frac{1}{2}\ln y + C'$$

$$= \ln \frac{1}{\sqrt{y}} + C'$$

$$\Rightarrow x = A\frac{1}{\sqrt{y}} \quad \text{or} \quad y = \frac{A}{x^2}$$



c)

$$\oint u \cdot dx = \iint \omega_z \, dA = \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA =$$

$$= \iint -2\alpha y \, dx dy = \left\{ \begin{matrix} \text{by} \\ \text{symmetry} \end{matrix} \right\} = 0$$

$\boxed{2.}$

a) Rate of change of $\underset{\sim}{u}$ at fixed point

$$\frac{\partial \underset{\sim}{u}}{\partial t} = \frac{d}{dt}\alpha x^2 \underset{\sim}{e_x} - 2\frac{d\alpha}{dt} xy \underset{\sim}{e_y}$$

$$= \{at(L,L)\} = \frac{d}{dt}\alpha L^2 \underset{\sim}{e_x} - 2\frac{d\alpha}{dt} L^2 \underset{\sim}{e_y}$$

b) Rate of change of $\underset{\sim}{u}$ for fluid element

$$\frac{D}{Dt}\underset{\sim}{u} = \frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u}\cdot\nabla \underset{\sim}{u}$$

$$\underset{\sim}{e_x}: \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \underbrace{v\frac{\partial u}{\partial y}}_{=0} = \frac{d\alpha}{dt}x^2 + \alpha x^2 \alpha 2x$$

$$= \{at(L,L)\} = \frac{d\alpha}{dt}L^2 + \alpha^2(t)\,2L^3$$

$$\underset{\sim}{e_y}: \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -2\frac{d\alpha}{dt}xy + \alpha x^2(-$$

$$= -2\frac{d\alpha}{dt}xy + \alpha x^2(-2\alpha y) - 2\alpha xy(-2x\alpha) =$$

$$= \{at\ (L,L)\} = -2\frac{d\alpha}{dt}L^2 - 2\alpha^2 L^3 + 4\alpha^2 L^3$$

$$= -2\frac{d\alpha}{dt}L^2 + 2\alpha^2 L^3$$

2c)

$$\frac{D\underset{\sim}{u}}{Dt_{obs}} = \frac{\partial \underset{\sim}{u}}{\partial t} + U_{obs}\frac{\partial \underset{\sim}{u}}{\partial x} =$$

$$= \frac{d\alpha}{dt}\left(x^2 \underset{\sim}{e_x} - 2xy\,\underset{\sim}{e_y}\right) +$$

$$+ U_{obs}\,\alpha(t)\left(2x\,\underset{\sim}{e_x} - 2y\,\underset{\sim}{e_y}\right) =$$

$$= \left\{ \text{at } (L,L) \right\} = \frac{d\alpha}{dt}\left(L^2 \underset{\sim}{e_x} - 2L^2\,\underset{\sim}{e_y}\right) +$$

$$+ U_{obs}\,\alpha(t)\,2L\left(\underset{\sim}{e_x} - \underset{\sim}{e_y}\right)$$

$\boxed{3.}$

a) Flow rate $Q_B = U_0 \cdot \pi R_0^2$

since volume is conserved for an incompressible fluid.

b) See recitations Exercise 5 example 3.

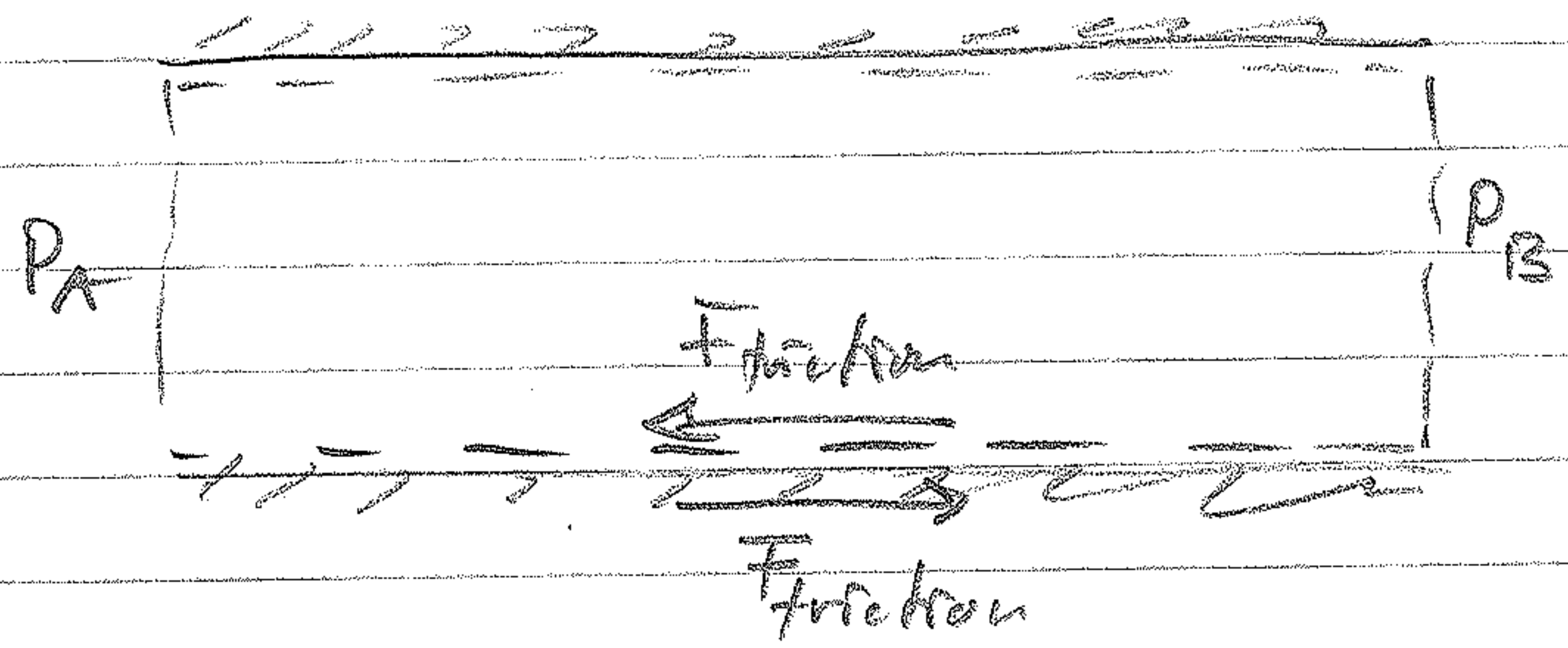
then $u_z = -\dfrac{1}{4\mu}\dfrac{\partial p}{\partial z}\left(r_0^2 - r^2\right)$

Flow rate $Q_B = \displaystyle\int_0^{r_0} u_z\, 2\pi r\, dr = -\frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{r_0} 2\pi\left(r_0^2 r - r^3\right)dr =$

$$= -\frac{1}{4\mu}\frac{\partial p}{\partial z}\left[2\pi\left(r_0^2\frac{r^2}{2}-\frac{r^4}{4}\right)\right]_0^{r_0} = -\frac{\pi}{8\mu}\frac{\partial p}{\partial z}r_0^4$$

$$p_A - p_B = -\frac{\partial p}{\partial z}\cdot L = \underline{\underline{Q_B L \frac{8\mu}{\pi r_0^4}}}$$

c)



Controlvolume momentum balance

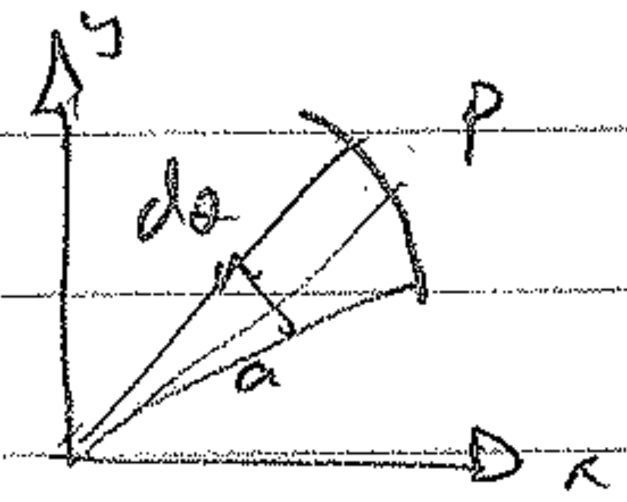
$\rightarrow$:

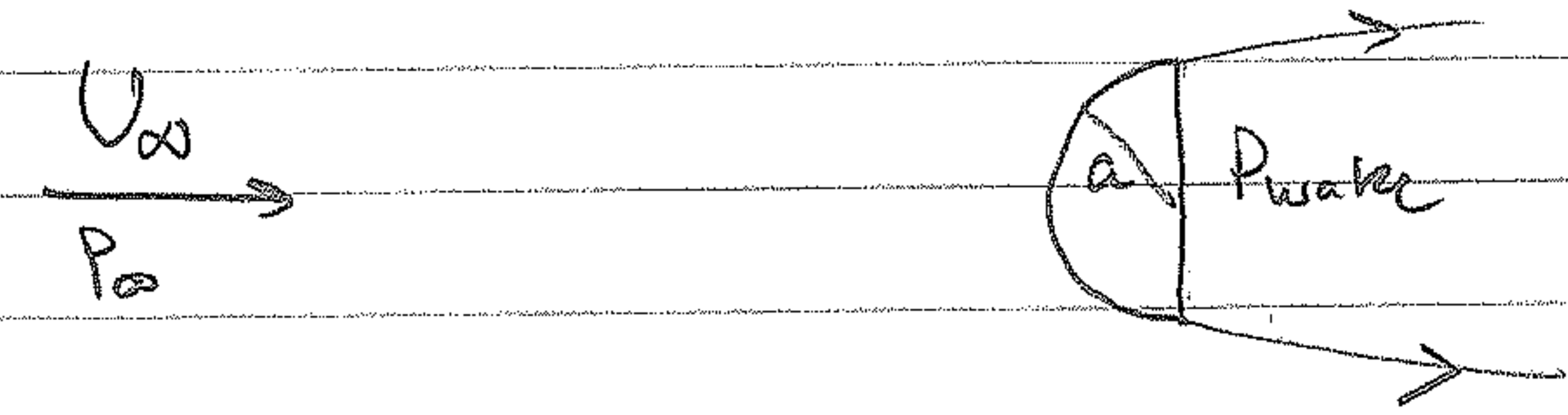
$$0 = (P_A - P_B)\pi r_0^2 - F_{friction}$$

$$\Rightarrow F_{friction} = \cancel{\pi} r_0^2 Q_B L \frac{8\mu}{\cancel{\pi} r_0^4} = \underline{\underline{Q_B \frac{L 8\mu}{r_0^2}}}$$

4.

a)





$$\frac{D'}{2} = \int_{\pi/2}^{\pi} -(p-p_\infty)\, a\cos\theta\, d\theta - (p_{wake}-p_\infty)\, a =$$

$$= \int_{\pi/2}^{\pi} -\tfrac{1}{2}\rho U_\infty^2 C_p\, a\cos\theta\, d\theta - \tfrac{1}{2}\rho U_\infty^2 C_p\, a =$$

$$= \tfrac{1}{2}\rho U_\infty^2 a \left[\int_{\pi/2}^{\pi} -(1-4\sin^2\theta)\cos\theta\, d\theta - (1-4\cdot 1)\right] =$$

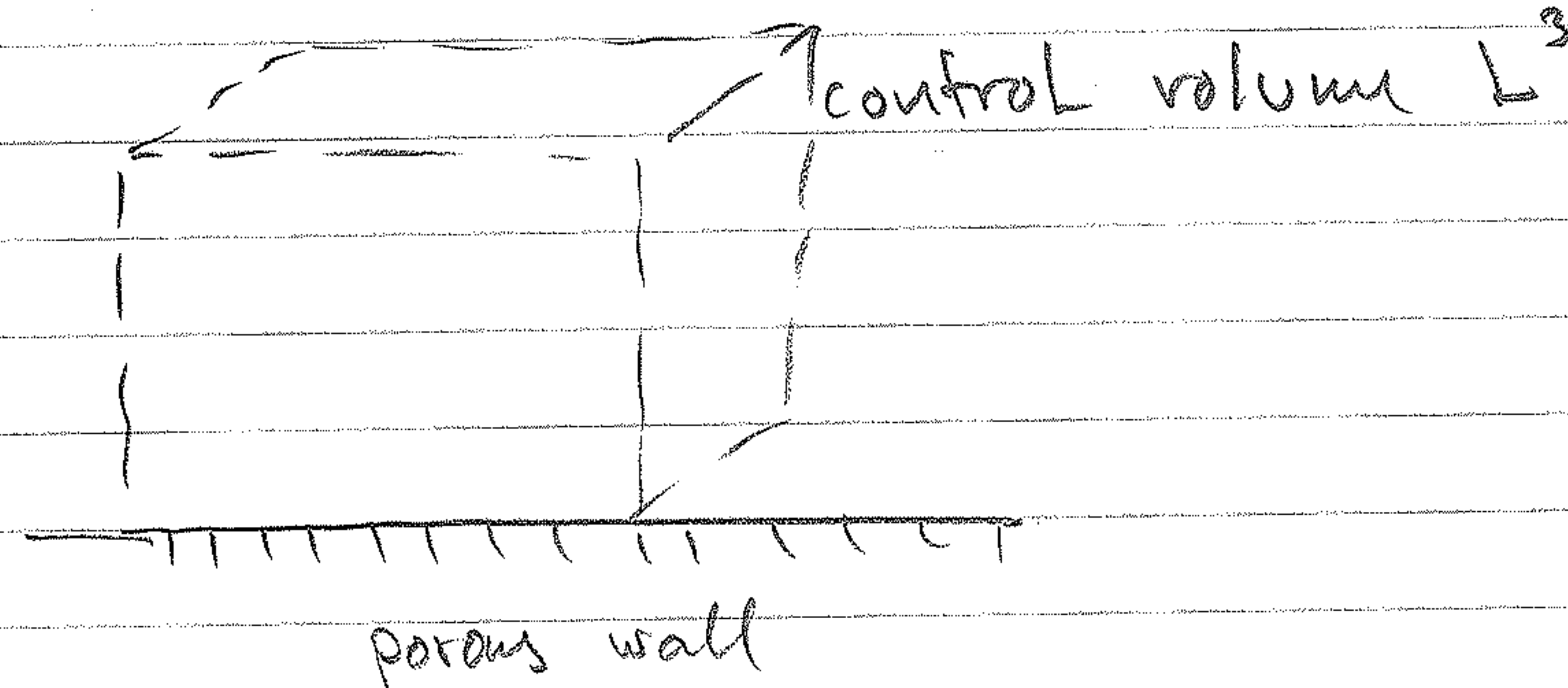
$$= \tfrac{1}{2}\rho U_\infty^2 a \left[1 + \underbrace{4\int_{\pi/2}^{\pi} \sin^2\theta\cos\theta\, d\theta}_{4\cdot\int_1^0 x^2 dx = 4\left(-\frac{1}{3}\right)} + 3\right] =$$

$$= \tfrac{1}{2}\rho U_\infty^2 a \; 4\left(1-\tfrac{1}{3}\right) = \tfrac{4}{3}\rho U_\infty^2 a$$

$$D' = \tfrac{8}{3}\rho U_\infty^2 a$$

4b) Generally, the boundary layer separates as the flow just outside the b.-l. decelerates. The criteria for the separation point is $\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0$ where y is the wall normal coordinate.

5.



$$\underbrace{\frac{d}{dt}\int_V \rho\left(e+\tfrac{1}{2}u_iu_i\right)dV}_{=0 \text{ steady state}} + \underbrace{\oint_S \rho\left(e+\tfrac{1}{2}u_iu_i\right)u_k n_k \, dS}_{1} = \underbrace{\int_V \rho g_i u_i \, dV}_{=0 \text{ neglected}} +$$

$$+ \underbrace{\oint_S u_i\left(-p\delta_{ik}+\tau_{ik}\right)n_k \, dS}_{\text{net is zero}} - \underbrace{\oint_S q_i n_i \, dS}_{2}$$

because of fully developed flow.

$$① = \rho\left(e_\infty + \frac{1}{2}U_\infty^2 + \frac{1}{2}V_\infty^2\right)(-V_\infty)L^2$$

$$+ \rho\left(e_{wall} + 0 + \frac{1}{2}V_\infty^2\right)(V_\infty)L^2 =$$

$$= \rho V_\infty L^2\left(e_{wall} - e_\infty - \frac{1}{2}U_\infty^2\right)$$

$$= \rho V_\infty L^2\left(C_p(T_{wall} - T_\infty) - \frac{1}{2}U_\infty^2\right)$$

$$② = \left(q_{y\infty} - q_{y\,wall}\right)L^2$$

$$q_y = -k\frac{\partial T}{\partial y} = -k\frac{V_\infty}{\nu}\Big[\Delta T(-P_r)e^{-\eta P_r}$$

$$+ \frac{U_\infty^2}{2C_p}\frac{P_r}{P_r-2}\left(-2e^{-2\eta} + P_r e^{-\eta P_r}\right)\Big]$$

$$q_{y\infty} = 0, \quad -q_{y\,wall} = k\frac{V_\infty}{\nu}\left[-\Delta T P_r + \frac{U_\infty^2}{2C_p}\frac{P_r}{\cancel{P_r-2}}\cancel{(-2+P_r)}\right]$$

$$= \frac{k V_\infty P_r}{\nu}\left[-\Delta T + \frac{U_\infty^2}{2C_p}\right]$$

$$= \frac{\cancel{k}V_\infty}{\nu}\frac{\mu}{\cancel{k}}\left[-\Delta T C_p + \frac{U_\infty^2}{2}\right]$$

$$② = \rho V_\infty L^2\left[-C_p(T_w - T_\infty) + \frac{U_\infty^2}{2}\right] = -①$$