Exam, SG2214 Fluid Mechanics
4 February 2012, at 09:00-13:00
Examiner: Anders Dahlkild (ad@mech.kth.se)

Copies of Cylindrical and spherical coordinates*, which will be supplied if necessary, can be used for the exam as well as a book of basic math formulas, a calculator and an English dictionary. The point value of each question is given in paranthesis and you need more than 20 points to pass the exam including the points obtained from the homework problems.*

1. (10 p.) a) Derive the velocity profile for a pressure driven fully developed laminar viscous flow of an incompressible fluid in a horizontal pipe of circular cross section if the mean velocity over the cross section, $\overline{U}$, is specified. b) Obtain the corresponding pressure drop required along a length L of the pipe. c) Calculate the momentum flux through a cross section of the pipe.

2. (10 p.) Estimate the diameter of a 2D cylinder that experience the same drag per unit width as a 5 meter long flat plate aligned with the flow at $U_\infty = 5$ m/s in air at 20°C. Necessary data are given in figure 1. Make sketches of the two flow fields in question and discuss the concepts of pressure drag and viscous drag in relation to the two cases illustrated.

3. (10 p.) The complex potential of an irrotational 2D flow field is given by

$$F(z) = U_\infty(z + b\ln z).$$

a) Obtain the vector field for the velocity, $\mathbf{u}$, and find possible stagnation points. Make a rough sketch of the streamlines.
b) Obtain the pressure field along the x-axis (y=0).
c) Obtain the acceleration vector, $\mathbf{a}$, of a material fluid element along the x-axis.

4. (10 p.) Consider a two-dimensional jet, which is an efflux of fluid from a long and narrow orifice, submerged in the same fluid as that of the jet itself. See also figure 2.
a) Under what circumstances can we use the boundary layer approximation to study the jet?
b) Write down the boundary layer equations for flows with zero pressure gradient.
c) The stream function for the laminar two-dimensional jet can be written

$$\Psi = \sqrt{\nu x u_0(x)}f(\eta), \quad \eta = y/\sqrt{\nu x/u_0(x)},$$

where $u_0(x) = Cx^{-1/3}$ is the centre line velocity in the jet and C is a dimensional constant. Obtain expressions for $u(x, y)$ and $v(x, y)$ under the assumption that f is a known function of η.

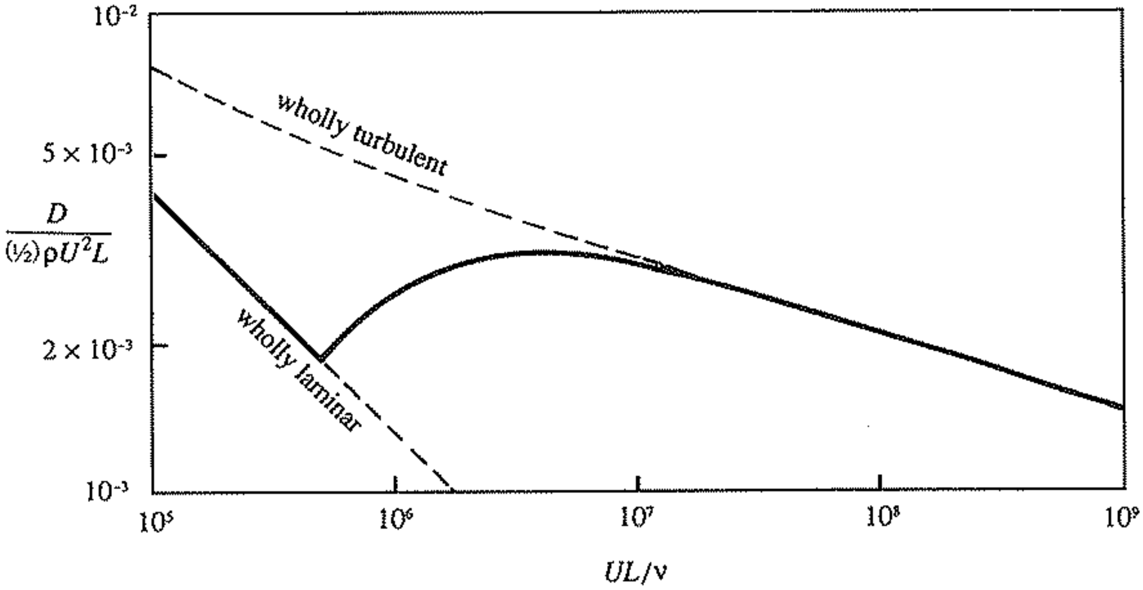
5. (10 p.) Consider fully developed laminar plane Couette flow between two parallel, horizontal plates separated by a distance h. Let the difference in speed between the plates be U. The plates are kept at constant temperature T_w. The fluid may be assumed incompressible with density ρ, specific heat c_p, constant viscosity μ and coefficient of heat conduction k. Take a rectangular control volume, V, for which the two horizontal surfaces, $L \times L$, coincide with the walls and the two vertical surfaces, $L \times h$, are perpendicular to the flow. Evaluate each term of the energy equation below for this control volume.

$$\frac{d}{dt}\int_V \rho c_p T dV + \oint_S \rho c_p T(\mathbf{u}\cdot\mathbf{n})dS = -\int_V p\nabla\cdot\mathbf{u}\, dV - \oint_S \mathbf{q}\cdot\mathbf{n}\, dS + \int_V \Phi dV$$

$$\mathbf{q} = -k\nabla T, \quad \Phi = 2\mu\overline{e}_{ij}\overline{e}_{ij} + \mu_B(\frac{\partial u_l}{\partial x_l})^2, \quad \overline{e}_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_i}{\partial x_j}) - \frac{\delta_{ij}}{3}\frac{\partial u_l}{\partial x_l}$$

A3. Properties of Dry Air at Atmospheric Pressure

T °C	ρ kg/m^3	μ kg m^{-1} s^{-1}	ν m^2/s	κ m^2/s	Pr ν/κ
0	1.293	1.71E − 5	1.33E − 5	1.84E − 5	0.72
10	1.247	1.76E − 5	1.41E − 5	1.96E − 5	0.72
20	1.200	1.81E − 5	1.50E − 5	2.08E − 5	0.72
30	1.165	1.86E − 5	1.60E − 5	2.25E − 5	0.71
40	1.127	1.87E − 5	1.66E − 5	2.38E − 5	0.71
60	1.060	1.97E − 5	1.86E − 5	2.65E − 5	0.71
80	1.000	2.07E − 5	2.07E − 5	2.99E − 5	0.70
100	0.946	2.17E − 5	2.29E − 5	3.28E − 5	0.70



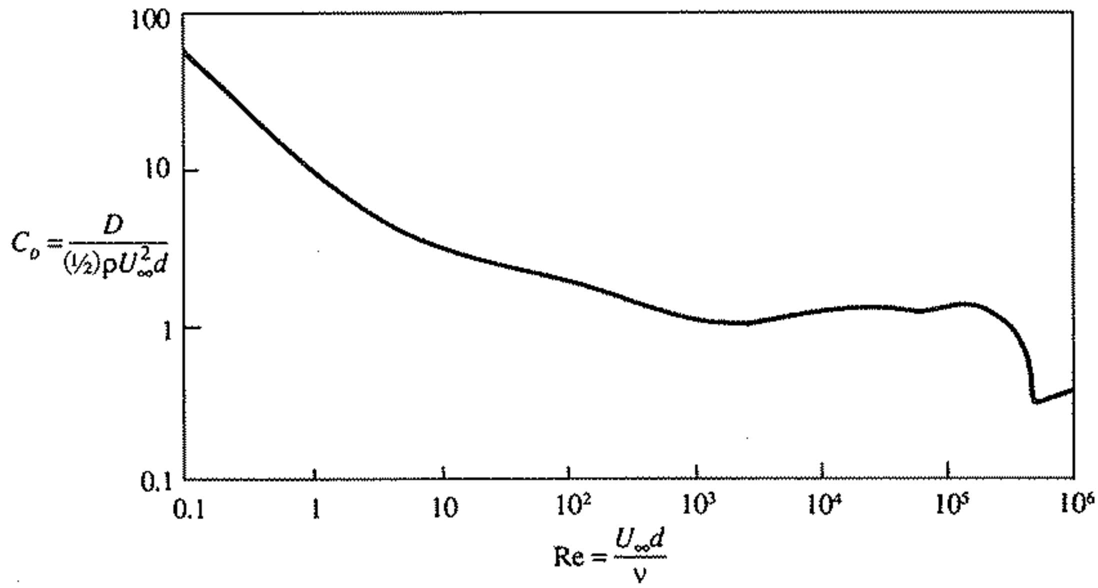


Figure 1. Drag coefficients for flat plate and circular cylinder. (From Kundu and Cohen 2008, Fluid Mechanics, 4:th ed.)

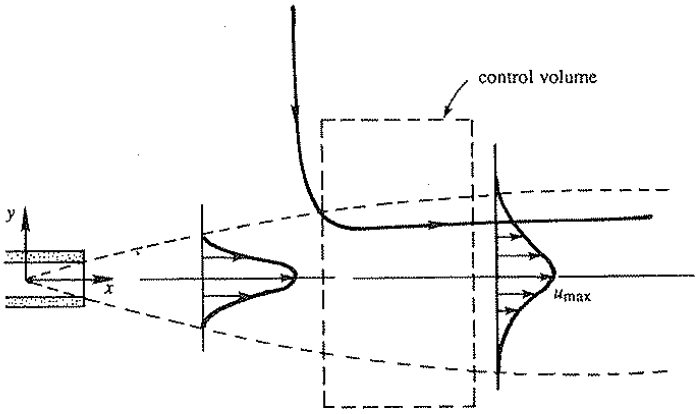


FIGURE 2. Two-dimensional jet. (From Kundu and Cohen 2008, Fluid Mechanics, 4:th ed.)

Exam 2012-02-04 Fluid Mechanics SG2214

1.
$$0 = -\frac{dp}{dx} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u\right)$$
for fully developed pipe flow.

Boundary condition: $u(r=R)=0$

Mean velocity $\bar{U} = \frac{Q}{\pi R^2} = \frac{1}{\pi R^2}\int_0^R u\,2\pi r\,dr$

Integration $\Rightarrow$ $\frac{\partial u}{\partial r} = \frac{1}{\mu}\frac{dp}{dx}\frac{r}{2} + \frac{C}{r}$

$$\Rightarrow \quad u(r) = \frac{1}{\mu}\frac{dp}{dx}\frac{r^2}{4} + C\ln r + D$$
$C=0$

B.C. $\Rightarrow D = -\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{4}$

$$u(r) = -\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{4}\left(1-\left(\frac{r}{R}\right)^2\right) = \underline{2\bar{U}\left(1-\left(\frac{r}{R}\right)^2\right)}$$ a) (5p)

$$\bar{U} = \frac{1}{\cancel{\pi} R^2}\int_0^R 2\cancel{\pi} r\left(1-(r/R)^2\right)dr\left(-\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{4}\right) =$$

$$= -\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{2}\int_0^1 x(1-x^2)\,dx = -\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = -\frac{1}{\mu}\frac{dp}{dx}\frac{R^2}{8}$$

$$\underline{\frac{dp}{dx} = -\frac{8\mu\bar{U}}{R^2}}$$ b) 2p.

c) Momentum flux $M = \int_0^R \varrho u^2 2\pi r dr =$

$$= \int_0^R \varrho 4\overline{U}^2 \left(1-(r/R)^2\right)^2 2\pi r dr = 8\pi \varrho R^2 \overline{U}^2 \int_0^1 (1-x^2)^2 x dx$$

$$= 8\pi \varrho R^2 \overline{U}^2 \int_0^1 (1-y)^2 \frac{dy}{2} = 8\pi \varrho R^2 \overline{U}^2 \underbrace{\left[-\frac{1}{2} \frac{(1-y)^3}{3} \right]_0^1}_{\frac{1}{6}} =$$

$$= \frac{4}{3} \pi \varrho R^2 \overline{U}^2 \quad (3p.)$$

(2.)

$$D'_0 = C_D \cdot \frac{1}{2} \rho U_\infty^2 \cdot d = D' = 2 \cdot C_F \frac{1}{2} \rho U_\infty^2 \cdot L$$

$$C_D \cdot d = 2 C_F \cdot L$$

$L = 5\ m$ $\quad Re_L = \frac{UL}{\nu} = \frac{5 \cdot 5}{1.5 \cdot 10^{-5}} = \frac{25 \cdot 2 \cdot 10^5}{3}$

$U_\infty = 5\ m/s$

$$= \frac{5}{3} \cdot 10^6 = 1.7 \cdot 10^6$$

Diagram $\Rightarrow$ $C_F \cong 3 \cdot 10^{-3}$

Suppose $C_D \sim 1$ $\Rightarrow$ $d = 2 \cdot 3 \cdot 10^{-3} \cdot 5\ m = 3 \cdot 10^{-2}\ m$

$$Re_d = \frac{U_\infty d}{\nu} = \frac{5 \cdot 3 \cdot 10^{-2}}{1.5\ 10^{-5}} = \frac{5 \cdot 3 \cdot 2 \cdot 10^3}{3} = 10^4$$

$Re_d = 10^4 \rightarrow$ diagram $\rightarrow C_D \sim 1$ OK

$\therefore$ $\underline{d \cong 3\ cm}$

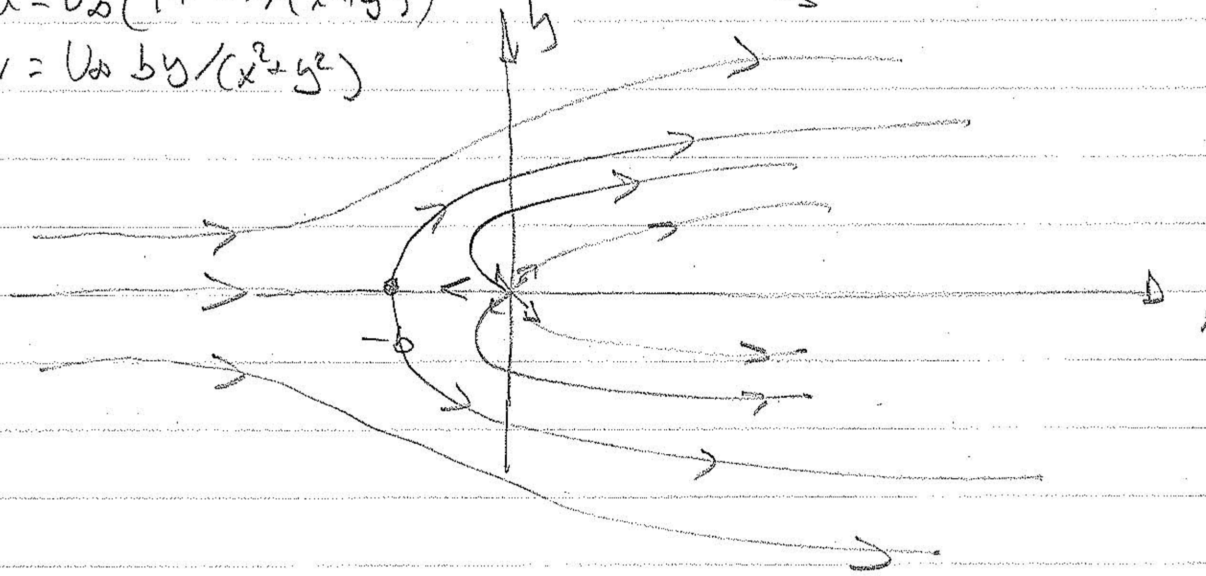
(3.) $F(z) = U_\infty(z + b\ln z)$

a) $F'(z) = u - iv = U_\infty(1 + b/z) \Leftrightarrow \overset{U_\infty}{\rightrightarrows} + $ $m = 2\pi b$

(5) $= U_\infty\left(1 + \frac{b z^*}{|z|^2}\right) = U_\infty\left(1 + \frac{b(x-iy)}{x^2+y^2}\right)$

$u - iv = 0 \Rightarrow U_\infty\left(1 + \frac{b}{z_s}\right) = 0 \Rightarrow z_s = -b$

$$\begin{cases} u = U_\infty\left(1 + bx/(x^2+y^2)\right) \\ v = U_\infty\, by/(x^2+y^2) \end{cases}$$



(3) b) Irrotational flow $\Rightarrow$ Bernoulli eq. valid

$\Rightarrow p_\infty + \frac{1}{2}\rho U_\infty^2 = p + \frac{1}{2}\rho(u^2+v^2)$

Along x-axis $v = 0$; $u = U_\infty\left(1 + \frac{b}{x}\right)$

$p - p_\infty = \frac{1}{2}\rho U_\infty^2 - \frac{1}{2}\rho U_\infty^2\left(1 + \frac{b}{x}\right)^2$

$= \frac{1}{2}\rho U_\infty^2\left(1 - (1 + b/x)^2\right)$

(2) c) $\vec{a} = \frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p = \frac{-1}{\cancel{\rho}}\nabla\left(-(1+b/x)^2\right)\frac{1}{2}\cancel{\rho} U_\infty^2$

$= U_\infty^2(1 + b/x)\left(-\frac{b}{x^2}\right)\vec{e_x} = -U_\infty^2 \frac{b(1+b/x)}{x^2}\vec{e_x}$

a) alt.

$$F' = U_\infty\left(1 + \frac{b}{z}\right) = U_\infty\left(1 + \frac{b}{r} e^{-i\theta}\right) =$$

$$= U_\infty\left(e^{i\theta} + \frac{b}{r}\right) e^{-i\theta} = U_\infty\left(\cos\theta + i\sin\theta + \frac{b}{r}\right)$$

$$= U_\infty\left(\cos\theta + \frac{b}{r} - i(-\sin\theta)\right) = (u_r - iu_\theta)\, e^{-i\theta}$$

$$\begin{cases} u_r = U_\infty\left(\cos\theta + \frac{b}{r}\right) \\ u_\theta = -U_\infty \sin\theta \end{cases}$$

④

a) $Re_x = \frac{\rho u_0(x)\, x}{\mu} \gg 1$

or ⟺ jet width ≪ jet length.

b) $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$; $\frac{\partial p}{\partial y} = 0$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

c) $u(x,y) = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left[\sqrt{\nu x u_0(x)}\, f(\eta)\right] = \sqrt{\ } f' \frac{\partial \eta}{\partial y} =$

$= \sqrt{\ } f' \frac{1}{\sqrt{\nu x / u_0(x)}} = u_0(x) f'(\eta)$

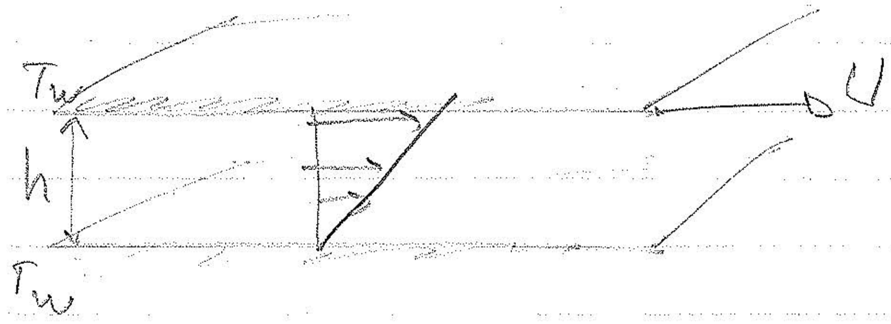
$v(x,y) = -\frac{\partial \psi}{\partial x} = -\frac{\partial}{\partial x}\left[\sqrt{\nu C}\, x^{1/3} f(\eta)\right] =$

$= -\sqrt{\nu C}\,\frac{1}{3} x^{-2/3} f(\eta) - \sqrt{\nu C}\, x^{1/3} f'(\eta)\frac{\partial \eta}{\partial x} = \left\{\frac{\partial \eta}{\partial x} = -\frac{2}{3}\frac{\eta}{x}\right\} =$

$= -\sqrt{\nu C}\,\frac{1}{3} x^{-2/3} f(\eta) + \sqrt{\nu C}\, x^{-2/3}\frac{2}{3}\eta f'(\eta)$

$= -u_0(x)\sqrt{\frac{\nu}{u_0(x)\, x}}\,\frac{1}{3}\left[f - 2\eta f'\right]$

(5)



$u = U\frac{y}{h}$ $\quad v = 0 \quad$ $\Phi = \mu\left(\frac{du}{dy}\right)^2 = \mu\left(\frac{U}{h}\right)^2$

$\frac{\partial u}{\partial x} = 0$

Fully developed flow $\Rightarrow \frac{\partial}{\partial x} = 0 \quad \frac{\partial}{\partial t} = 0$

$\frac{d}{dt}\int \rho c_p T\,dV = 0$ since steady state

$\oint_S \rho c_p T\,\vec{u}\cdot\vec{n}\,dS = 0$ since either $\vec{u}\cdot\vec{n} = 0$

or inflow = outflow since flow is fully developed.

$-\int p\nabla\cdot\underline{u}\,dV = 0$ since $\nabla\cdot\vec{u} = 0$

$-\oint_S \vec{q}\cdot\vec{n} = -L^2\left(-k\frac{\partial T}{\partial y}(y=h) + k\frac{\partial}{\partial y}T(y=0)\right)$

$= -L^2 k\left[\frac{\partial T}{\partial y}(y=0) - \frac{\partial T}{\partial y}(y=h)\right] = -L^2\,2k\,\frac{\partial T}{\partial y}(y=0)$

$\int_V \Phi\,dV = L^2\int_0^h \mu\left(\frac{du}{dy}\right)^2 dy = L^2\mu\left(\frac{U}{h}\right)^2\cdot h = L^2\frac{\mu U^2}{h}$

Also, $0 = k \frac{d^2T}{dy^2} + \mu\left(\frac{U}{h}\right)^2$

$$T = -\frac{\mu}{k}\left(\frac{U}{h}\right)^2 \frac{y^2}{2} + Ay + B$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\left(\frac{U}{h}\right)^2 y + A \qquad A = \frac{\mu}{k}\left(\frac{U}{h}\right)^2 \frac{h}{2} \qquad B = T_w$$

$$T = T_w + \frac{\mu}{k}\left(\frac{U}{h}\right)^2\left(\frac{hy}{2} - \frac{y^2}{2}\right)$$

$$= T_w + \frac{\mu}{k}\frac{U^2}{2}\left(\frac{y}{h} - \left(\frac{y}{h}\right)^2\right)$$

$$\frac{dT}{dy} = \frac{\mu}{k}\frac{U^2}{2}\left(\frac{1}{h} - \frac{2y}{h^2}\right)$$

$$\left.\frac{dT}{dy}\right|_{y=0} = \frac{\mu}{k}\frac{U^2}{2h} \Rightarrow L^2\, 2h\, \frac{\partial T}{\partial y}(y=0) = L^2 \frac{\mu U^2}{h}$$