Exam, SG2214 Fluid Mechanics
20 October 2011, at 9:00-13:00
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Copies of Cylindrical and spherical coordinates*, which will be supplied if necessary, can be used for the exam as well as a book of basic math formulas, a calculator and an English dictionary. The point value of each question is given in paranthesis and you need more than 20 points to pass the exam including the points obtained from the homework problems.*

1. (10 p.) Consider the flow field given in plane polar coordinates by

$$u_r = 0, \quad u_\theta = \Omega r + \frac{2\dot{\gamma}a^2}{r},$$

where a is a given length and Ω and $\dot{\gamma}$ are constants [1/s].

a) Show that the velocity in Cartesian coordinates can be written $\mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3}$ where

$$u_i(\mathbf{x}) = \epsilon_{i3k}x_k(\Omega + \frac{2\dot{\gamma}a^2}{x_k x_k - x_3^2}), \quad \mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}.$$

b) Calculate the deformation rate tensor $\frac{\partial u_i}{\partial x_j}$.

c) At the point $(x_1 = a, x_2 = a)$, separate the flow field in the local translation, rotation and pure deformation of a small fluid element. Draw a figure of how a small square around this point aligned with the Cartesian coordinate axes deforms. Do the same thing also at the point $(x_1 = a, x_2 = 0)$.

Hint: The relative motion is $du_i = \frac{\partial u_i}{\partial x_j}dx_j$, where dx_j is the small position vector relative to the point in question.

2. (10 p.) Use the Navier-Stokes equations to derive the Bernoulli equation for unsteady, incompressible, irrotational flow. Also show, using tensor notation, that $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla(\frac{\mathbf{u}\cdot\mathbf{u}}{2}) - \mathbf{u}\times\omega$, where ω is the curl of the velocity vector.

3. (10 p.) Consider a long circular cylinder of radius r_0 rotating with angular velocity Ω in an infinite body of viscous fluid which is at rest infinitely far away from the body.

a) Find the velocity distribution at steady state.

b) Find the pressure field. At large distances from the cylinder it is known that $p(r \to \infty) = p_\infty - \rho g z$, where z is the coordinate along the vertical axis of symmetry of the cylinder and gravity, g, is in the negative $z-$ direction. Give an equation for a surface of constant pressure $p = p_\infty$.

c) Find the direction and magnitude of the shear stress at the cylinder surface.

d) What is the rate of work performed on the fluid (per unit length of the cylinder) at steady state?

e) What is the total rate of dissipation in the fluid (per unit length of the cylinder) at steady state?

4. (10 p.) Irrotational flow into a plane contraction is given by

$$u_r = -\frac{|m|}{2\pi r}, \quad u_\theta = 0.$$

a) Calculate the acceleration vector, $\mathbf{a}$, of a material fluid particle and pressure gradient, ∇p, along the plane $y = 0$.

b) Consider the viscous boundary layer along the x-axis, $x > 0$. Assume the stream function in the boundary layer can be written

$$\Psi(x, y) = -U(x)\delta(x)f(\eta), \ \delta(x) = \sqrt{\frac{\nu x}{U(x)}}, \ U(x) = \frac{|m|}{2\pi x}, \ \eta = \frac{y}{\delta(x)}.$$

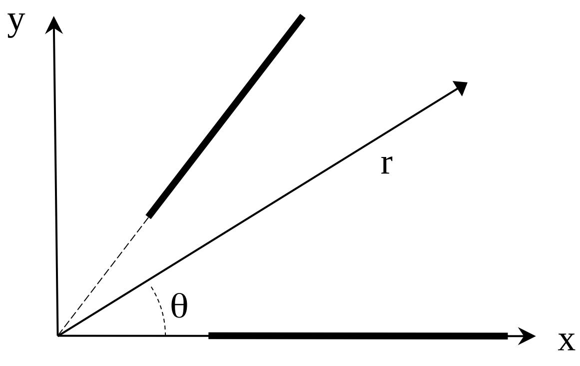

FIGURE 1. Plane contraction of opening angle α.

Derive expressions for $u(x, y)$ and $v(x, y)$ in terms of the functions above. *Hint*: Check the product $U(x)\delta(x)$.

c) Under what restrictions of the parameters can you speak of a thin boundary layer?

d) Derive an ordinary differential equation for $f(\eta)$ from the boundary layer equations. State also the necessary boundary conditions.

5. (10 p.) A layer of an incompressible viscous fluid of constant thickness flows due to gravity along an infinitely long inclined plate.

a) For given inclination of the plate, layer thickness, fluid viscosity and density, what is a relevant Reynolds number of the gravity driven, steady state flow?

b) In what limit of the Reynolds number, $Re >> 1$ or $Re << 1$ would one expect a turbulent flow?

c) Compare two fully developed flow cases along the plate where only the viscosities are different. One flow is laminar and the other is turbulent. In which case is the averaged wall shear stress larger? Or are they the same? Or is it not possible to say from the given information? You should motivate your answers.

d) The surrounding air is generally a quite poor conductor of heat. Estimate the temperature of the fluid at the surface in laminar flow if the temperature of the plate is T_0. Introduced variables and assumptions should be clearly defined.

Hint: $\rho c_p \frac{DT}{Dt} = \nabla \cdot (k\nabla T) + 2\mu e_{ij}e_{ij}$

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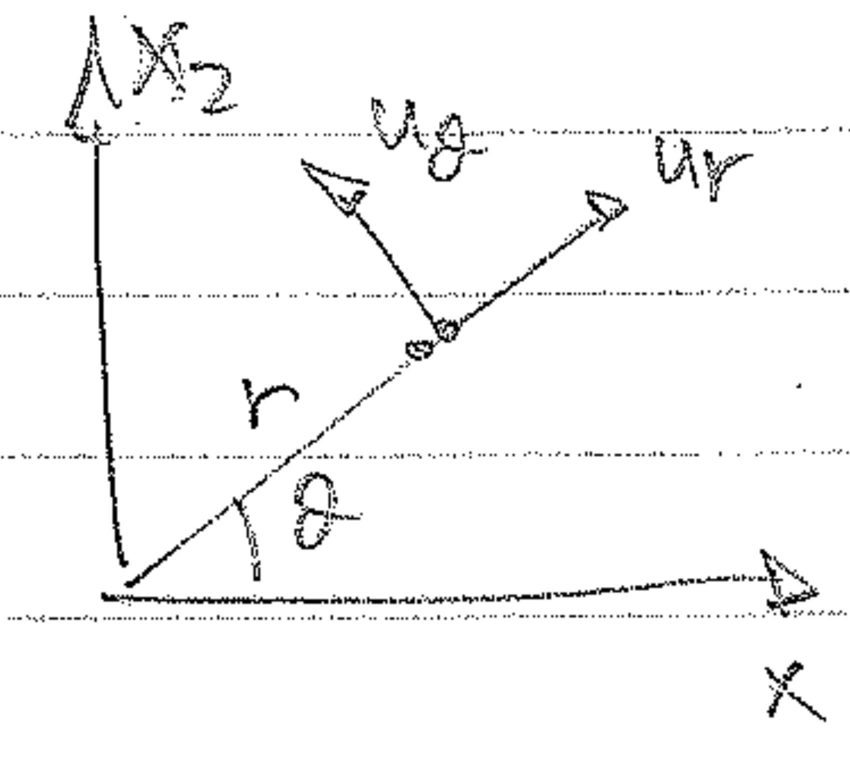
1. Cartesian vel. components

a)



$$u_1 = -u_\theta \sin\theta = -\left(\Omega r \sin\theta + 2\gamma a^2 \frac{\sin\theta}{r}\right)$$

$$u_2 = u_\theta \cos\theta = \Omega r \cos\theta + 2\gamma a^2 \frac{\cos\theta}{r}$$

Also $x_1 = r\cos\theta \quad x_2 = r\sin\theta$

$$\Rightarrow u_1 = -\left(\Omega x_2 + 2\gamma a^2 \frac{x_2}{r^2}\right)$$

$$u_2 = \Omega x_1 + 2\gamma a^2 \frac{x_1}{r^2} \qquad r^2 = x_1^2 + x_2^2$$

Using $u_i = \varepsilon_{i3k} x_k \left(\Omega + \frac{2\gamma a^2}{x_k x_k - x_3^2}\right)$ one obtains

$$u_1 = \left(\underbrace{\varepsilon_{131}}_{=0} x_1 + \underbrace{\varepsilon_{132}}_{=-1} x_2 + \underbrace{\varepsilon_{133}}_{=0} x_3\right)\left(\Omega + \frac{2\gamma a^2}{x_1^2 + x_2^2 + x_3^2 - x_3^2}\right)$$

$$= -\left(\Omega x_2 + \frac{2\gamma a^2 x_2}{x_1^2 + x_2^2}\right)$$

$$u_2 = \left(\underbrace{\varepsilon_{231}}_{=1} x_1 + \underbrace{\varepsilon_{232}}_{=0} x_2 + \underbrace{\varepsilon_{233}}_{=0} x_3\right)\left(\Omega + \frac{2\gamma a^2}{x_1^2 + x_2^2}\right)$$

$$= \Omega x_1 + \frac{2\gamma a^2 x_1}{x_1^2 + x_2^2}$$

b) $$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} \end{bmatrix}$$

$$\frac{\partial u_1}{\partial x_1} = \frac{4\dot{\gamma} a^2 x_1 x_2}{(x_1^2+x_2^2)^2} \qquad \frac{\partial u_1}{\partial x_2} = -\Omega + \frac{2\dot{\gamma} a^2 (x_2^2 - x_1^2)}{(x_1^2+x_2^2)^2}$$

$$\frac{\partial u_2}{\partial x_1} = \Omega + \frac{2\dot{\gamma} a^2 (x_2^2 - x_1^2)}{(x_1^2+x_2^2)^2} \qquad \frac{\partial u_2}{\partial x_2} = \frac{-4\dot{\gamma} a^2 x_1 x_2}{(x_1^2+x_2^2)^2}$$

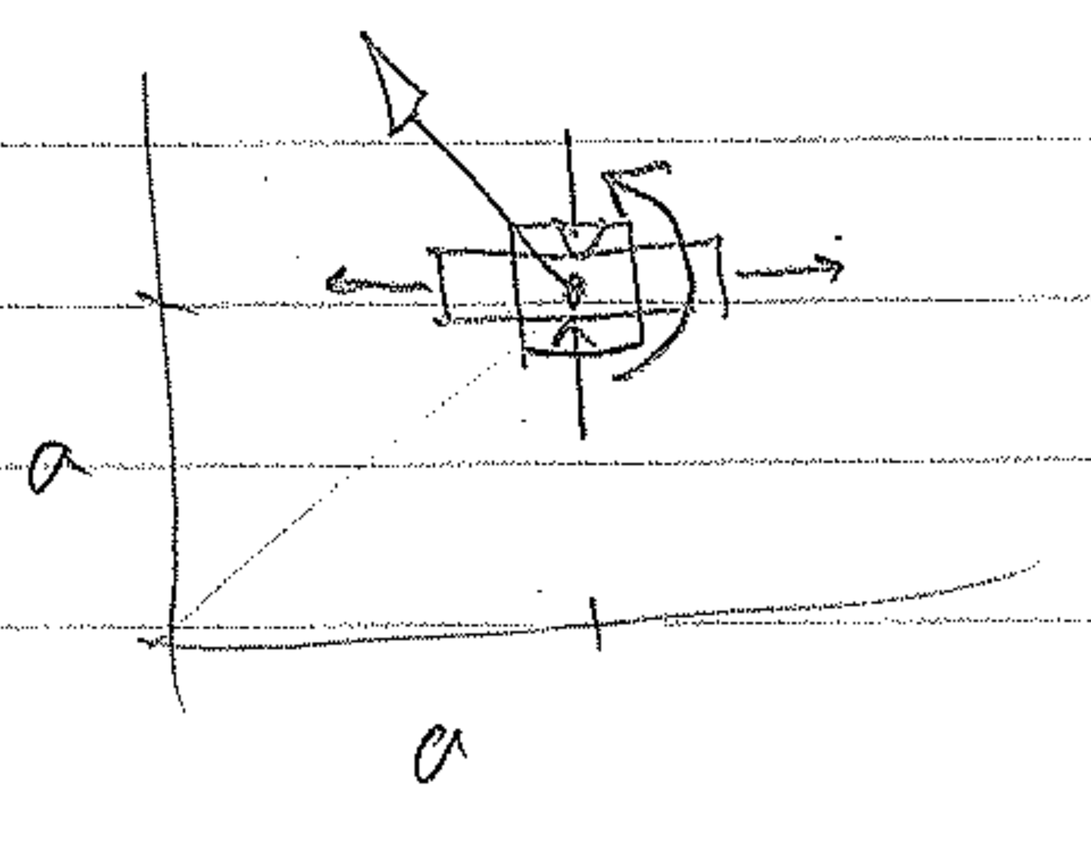
c) At $x_1 = x_2 = a$ translation $u_1 = -(\Omega a + \dot{\gamma} a)$

$u_2 = \Omega a + \dot{\gamma} a$

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} \dot{\gamma} & -\Omega \\ \Omega & -\dot{\gamma} \end{bmatrix} = \underbrace{\begin{bmatrix} \dot{\gamma} & 0 \\ 0 & -\dot{\gamma} \end{bmatrix}}_{\substack{e_{ij} \\ \text{strain rate}}} + \underbrace{\begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}}_{\substack{\xi_{ij} \\ \text{rotation rate}}}$$

$$du_{i\,def} = \begin{bmatrix} \dot{\gamma} & 0 \\ 0 & -\dot{\gamma} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} \dot{\gamma} dx_1 \\ -\dot{\gamma} dx_2 \end{bmatrix}$$

$$du_{i\,rot} = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -\Omega dx_2 \\ \Omega dx_1 \end{bmatrix}$$

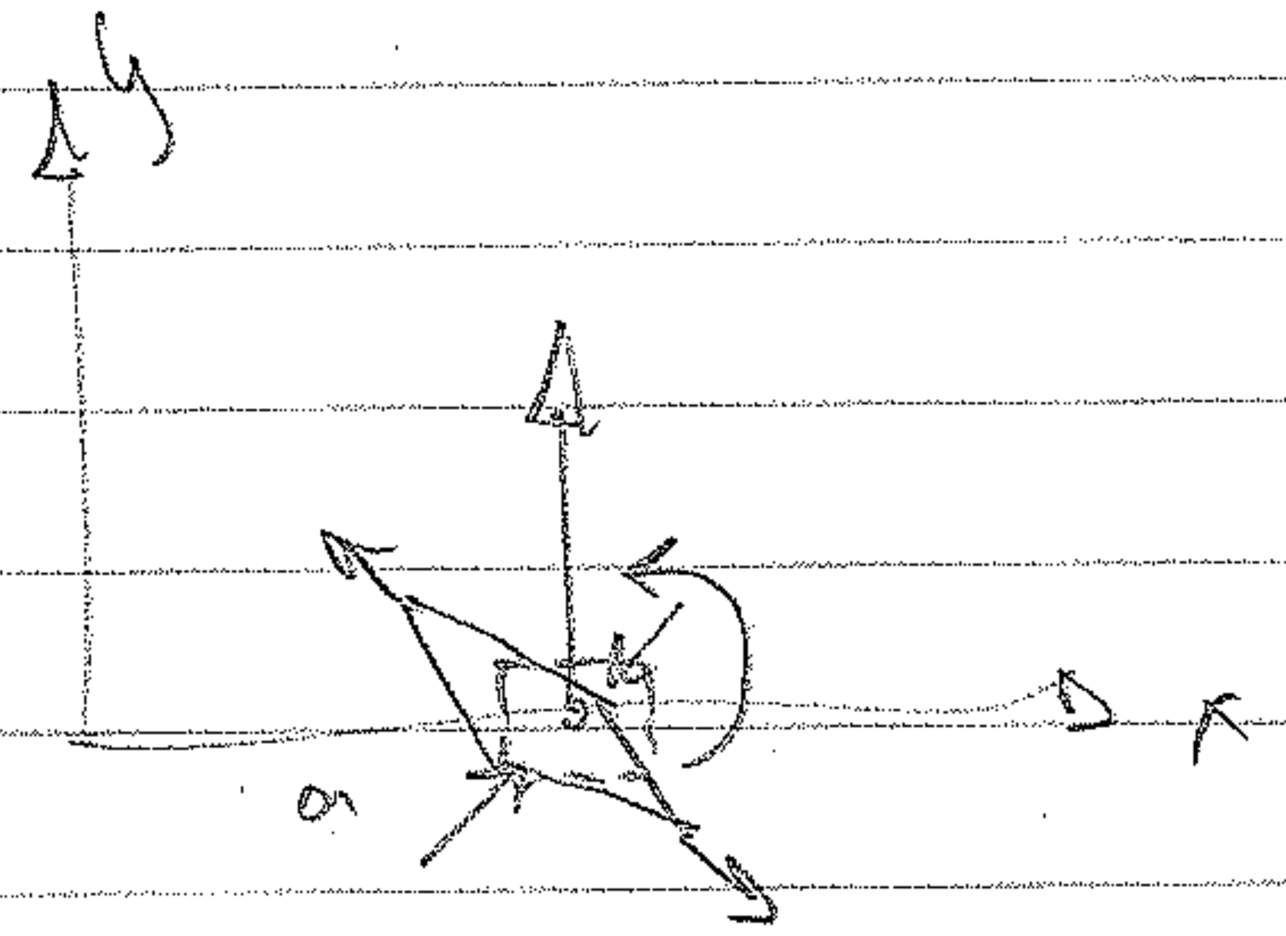


At $x_1 = a \quad x_2 = 0 \qquad \underline{u_1 = 0}, \ \underline{u_2 = \Omega a + 2\dot{\gamma} a}$

$$\frac{\partial u_i}{\partial x_j} = \begin{bmatrix} 0 & -\Omega - 2\dot{\gamma} \\ \Omega - 2\dot{\gamma} & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -2\dot{\gamma} \\ -2\dot{\gamma} & 0 \end{bmatrix}}_{\text{def.}} + \underbrace{\begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}}_{\text{rot.}}$$

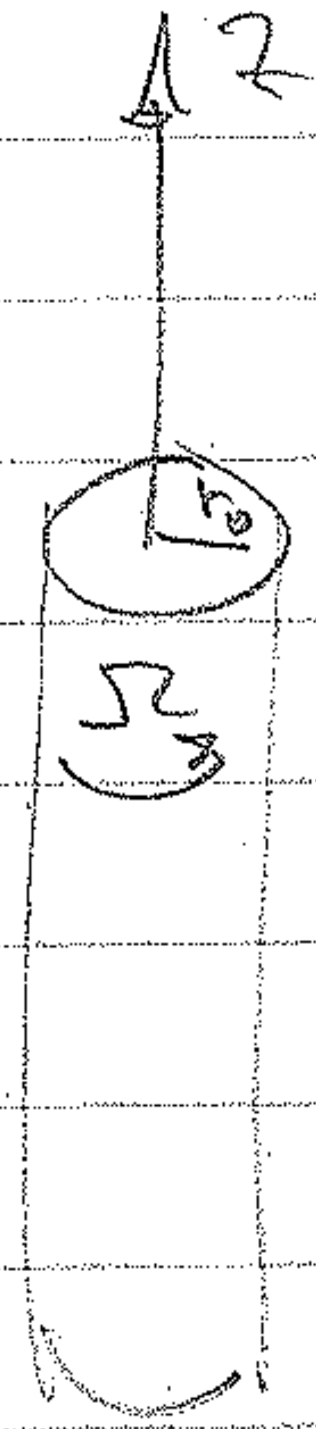
$$du_{i\,\text{def}} = \begin{bmatrix} 0 & -2\dot{\gamma} \\ -2\dot{\gamma} & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -2\dot{\gamma}\, dx_2 \\ -2\dot{\gamma}\, dx_1 \end{bmatrix}$$

$$du_{\text{rot}} = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} -\Omega dx_2 \\ \Omega dx_1 \end{bmatrix}$$



(2.) See lecture notes (use $\underset{\sim}{u} = \nabla\phi$).

(3.)



a) $u_\phi = \dfrac{\Omega r_0^2}{r}$ satisfies N-S eq.

b) Use N.-S. in r-direction

$$-\frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + 0$$

$$\frac{\partial p}{\partial r} = \frac{\rho u_\phi^2}{r} = \rho\frac{\Omega^2 r_0^4}{r^3}$$

$$p = -\frac{\rho\Omega^2 r_0^4}{2r^2} + f(z)$$

$$r \to \infty \Rightarrow f(z) = p_\infty - \rho g z$$

$$p(r,z) = p_\infty - \rho g z - \frac{1}{2}\rho\left(\frac{\Omega r_0^2}{r}\right)^2$$

Or, since $\nabla \times \underset{\sim}{u} = 0$, use Bernoulli's eq.

$$p + \frac{1}{2}\rho u_\phi^2 + \rho g z = \text{const.} = p_\infty$$

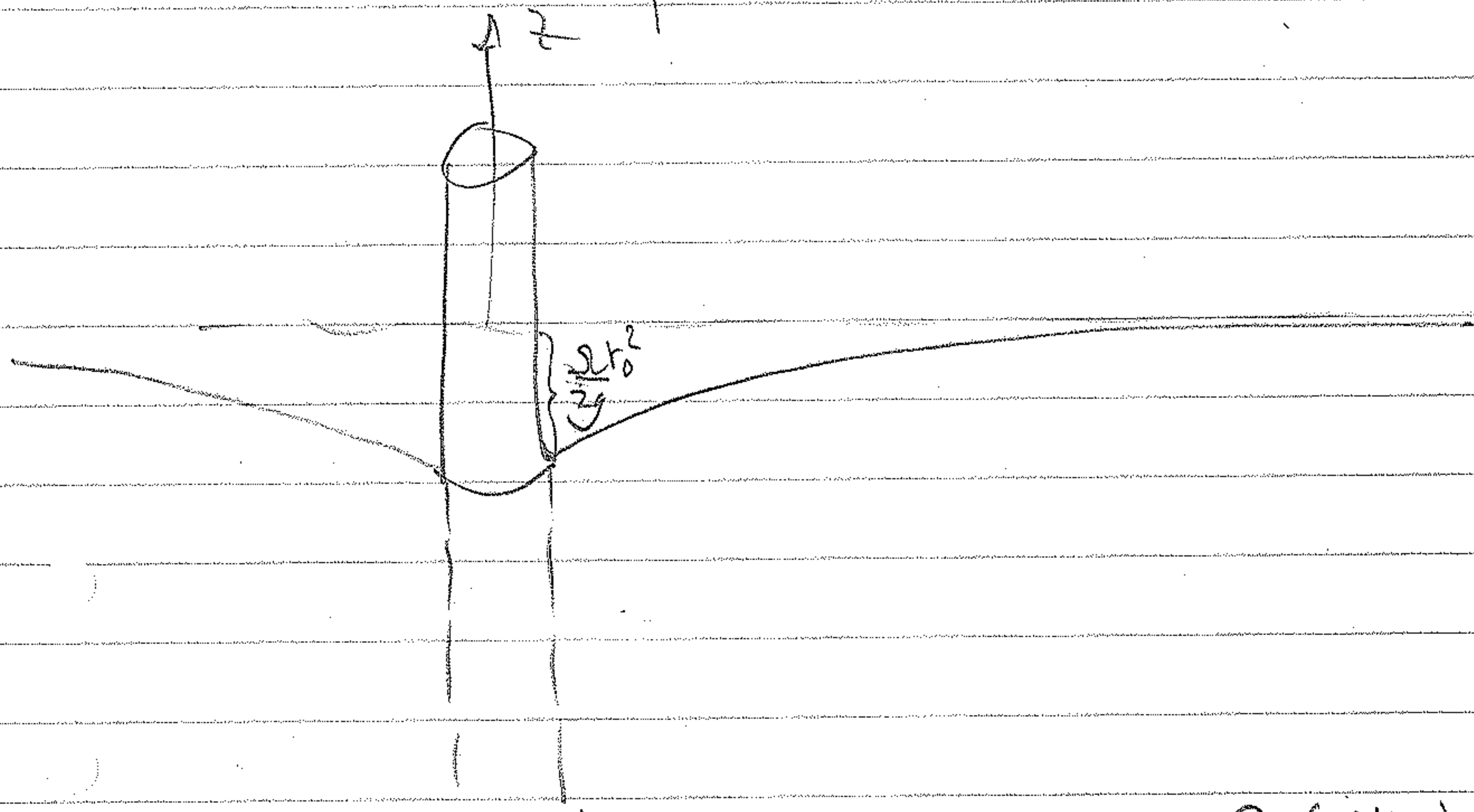
↑ same in the whole fluid for irrotational flow

Surface of constant pressure $p = p_\infty \Rightarrow$

$$\frac{1}{2}\rho\left(\frac{\Omega r_0^2}{r}\right)^2 + \rho g z = 0$$

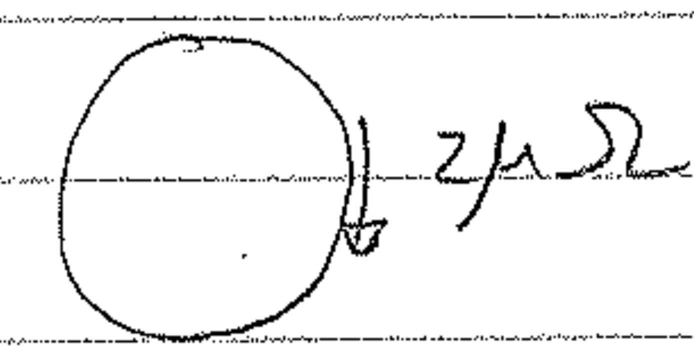
$$z(r) = -\frac{1}{2g}\frac{\Omega r_0^4}{r^2} \qquad z(r_0) = -\frac{1}{2g}\Omega r_0^2$$

Thus, if we had a free surface to the atmosphere at constant pressure it would have this form.



c) Shear stress $\tau_{r\theta} = 2\mu \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) =$

$$= 2\mu \frac{r}{2}\frac{\partial}{\partial r}\left(\frac{\Omega r_0^2}{r^2}\right) = -2\mu\frac{\Omega r_0^2}{r^2}$$



At $r = r_0$ $\tau_{r\theta_0} = -2\mu\Omega$

d) $\dot{W} = -\tau_{r\theta_0}\cdot 2\pi r_0 \cdot u_\theta(r_0) = 2\mu\Omega\, 2\pi r_0 \frac{\Omega r_0^2}{r_0} = 4\pi\mu(\Omega r_0)^2$

e) At steady state one must have dissip. $= \dot{W}$

(4)

a) At $y=0$ $\quad a_y = 0$

$$a_x = u\frac{\partial u}{\partial x} = u_r\frac{\partial u_r}{\partial r} = \frac{-|m|}{2\pi r}\frac{\partial}{\partial r}\left(\frac{-|m|}{2\pi r}\right) = \frac{-|m|}{2\pi r}\left(\frac{|m|}{2\pi r^2}\right)$$

$$= -\frac{|m|^2}{4\pi^2 x^3} \quad \frac{u_r^2}{r}$$

since $u\cdot\nabla u = -\frac{1}{\rho}\nabla p$

$\nabla p = -\rho\, u\cdot\nabla u = -\rho\, a$

b) $$u = \frac{\partial\psi}{\partial y} = -U\delta f'\frac{\partial\eta}{\partial y} = -U\delta f'\frac{1}{\delta} = -U f'(\eta)$$

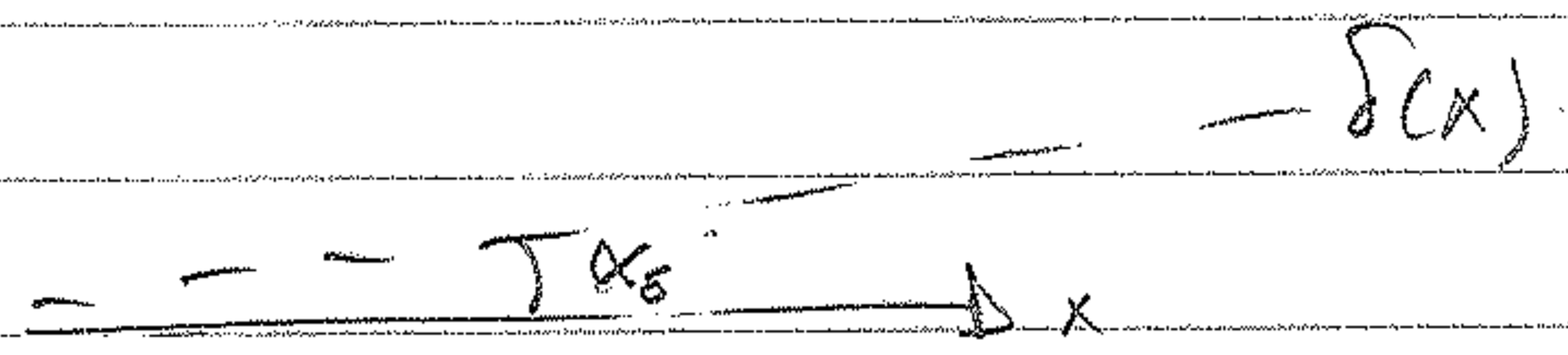
$$v = -\frac{\partial\psi}{\partial x} \quad ; \quad U(x)\delta(x) = \frac{|m|}{2\pi x}\sqrt{\frac{\nu x\, 2\pi x}{|m|}} = \sqrt{\frac{\nu|m|}{2\pi}} = \text{const.} \; !$$

$$v = -\frac{\partial\psi}{\partial x} = +U(x)\delta(x) f'\frac{\partial\eta}{\partial x} \quad ; \quad \frac{\partial\eta}{\partial x} = -\frac{\eta}{x}$$

$$= -U(x)\delta(x) f'\frac{\eta}{x}$$

c) $$\delta(x) = \sqrt{\frac{\nu x}{U(x)}} = \sqrt{\frac{\nu x\, 2\pi x}{|m|}} = x\sqrt{\frac{\nu 2\pi}{|m|}} = \frac{x}{\sqrt{Re}}$$

$$Re = \frac{|m|}{2\pi\nu}$$



Then if $\alpha_\delta \ll \alpha \Rightarrow \frac{\delta(x)}{x} = \frac{1}{\sqrt{Re}} \ll \alpha$

d) $$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$-\frac{1}{\varrho}\frac{dp}{dx} = U_e\frac{dU_e}{dx} = -U(x)\frac{d}{dx}\left(-U(x)\right) =$$

$$= -U(x)\frac{U(x)}{x} = -\frac{U^2(x)}{x}$$

$$\frac{\partial u}{\partial x} = -U'f' - Uf''\left(-\frac{\eta}{x}\right) = \frac{U}{x}f' + \frac{U}{x}\eta f''$$

$$= \frac{U}{x}\left(f' + f''\eta\right)$$

$$u\frac{\partial u}{\partial x} = -\frac{U^2}{x}f'\left(f' + \eta f''\right) \qquad \nu\frac{\partial^2 u}{\partial y^2} = -\frac{\nu U(x) f'''}{\delta^2} = -\frac{\nu U}{\nu x/U}f'''$$

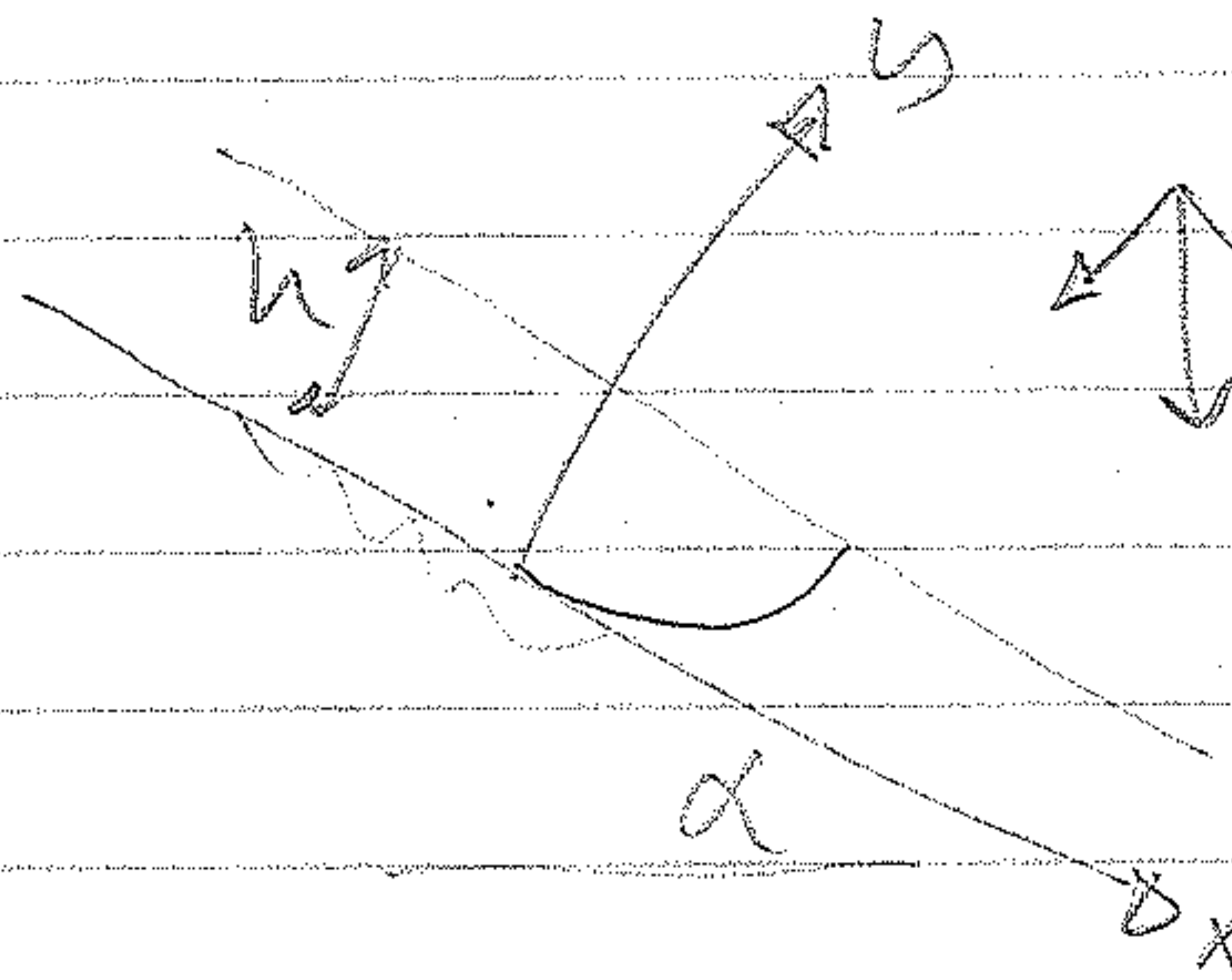
$$v\frac{\partial u}{\partial y} = -\frac{U(x)\delta(x)}{x}\eta f'\left(-U(x)\right)f''\frac{1}{\delta(x)} = \frac{U^2}{x}\eta f' f''$$

$$\Rightarrow -\frac{U^2}{x}f'\left(f' + \cancel{\eta f''}\right) + \cancel{\frac{U^2}{x}\eta f' f''} = -\frac{U^2}{x} - \frac{U^2}{x}f'''$$

$$\Rightarrow \boxed{f''' - (f')^2 + 1 = 0} \qquad f'(0) = 0 \quad f'(\infty) = 1$$

$$\left(f(0) = 0\right)$$

(5)



$g_x = g \sin\alpha$

a)

$$\frac{\partial p}{\partial x} = 0 \qquad \frac{\partial}{\partial x} = 0$$

$$0 = g_x + \nu \frac{\partial^2 u}{\partial y^2} \qquad u(0) = 0$$

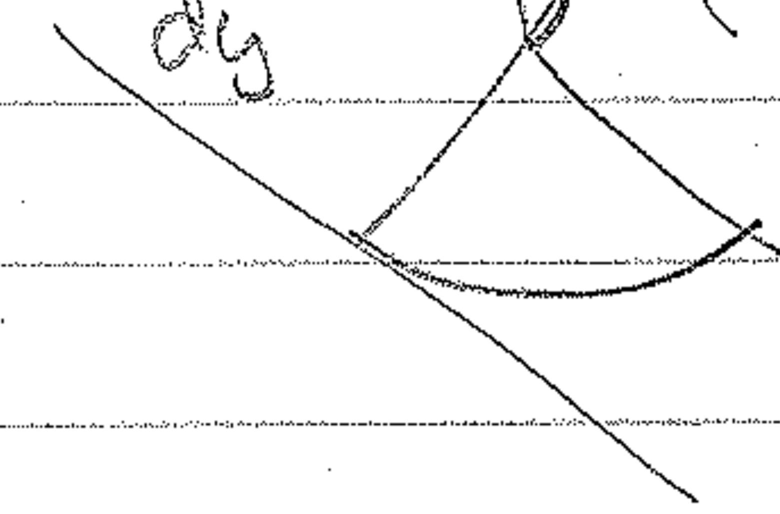
$$\frac{du}{dy}(h) = 0 \Rightarrow \tau_{xy}(h) = 0$$

$$\frac{d^2 u}{dy^2} = -\frac{g_x}{\nu} \qquad \frac{du}{dy} = -\frac{g_x y}{\nu} + C \qquad C = \frac{g_x h}{\nu}$$

$$u = -\frac{g_x y^2}{2\nu} + \frac{g_x h}{\nu} y + D \qquad D = 0$$

$$\frac{du}{dy} = \frac{g_x h}{\nu}\left(1 - \frac{y}{h}\right)$$

$$u = \frac{g_x h^2}{\nu} \frac{y}{h}\left(1 - \frac{1}{2}\frac{y}{h}\right)$$

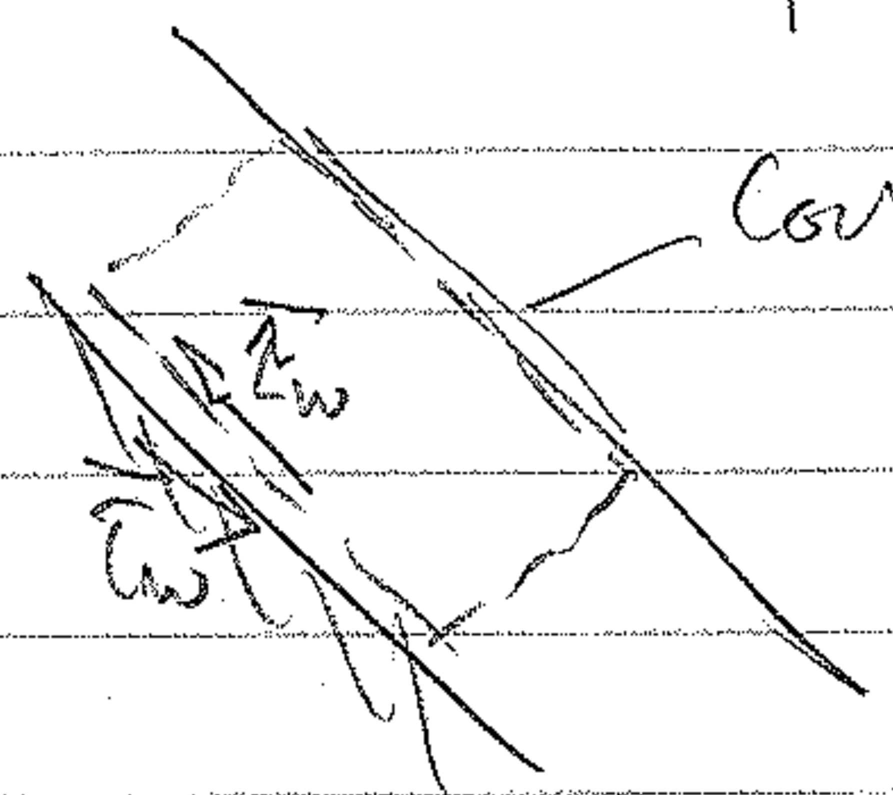


$$u_{max} = \frac{g_x h^2}{\nu} \frac{1}{2}$$

Relevant Re $\Rightarrow$ $Re = \frac{U_{max} h}{\nu} = \frac{g_x h^3}{2\nu^2}$

b) Turbulent if $Re \gg 1$

c)



Control volume

turbulent wall stress $\overline{\tau_w}$

momentum balance for fully dev. flow $\Rightarrow$ $\rho g_x h - \overline{\tau_w} = 0$

$\overline{\tau_w} = \rho g_x h$ indep. of viscosity, same as laminar case.

d) At surface then $q_y = -k \left.\frac{\partial T}{\partial y}\right|_{y=h} = 0$

$$0 = k \frac{d^2 T}{dy^2} + \mu \left(\frac{du}{dy}\right)^2$$

$$\frac{d^2 T}{dy^2} = -\frac{\mu}{k}\left(\frac{du}{dy}\right)^2 \; ; \; \left(\frac{du}{dy}\right)^2 = \left(\frac{g_x}{\nu}\right)^2 (h-y)^2$$

$$\frac{dT}{dy} = -\frac{\mu}{k}\left(\frac{g_x}{\nu}\right)^2 \left[-\frac{(h-y)^3}{3}\right] + \underset{\substack{\| \\ 0 \text{ since } \left.\frac{\partial T}{\partial y}\right|_{y=h} = 0}}{\text{const.}}$$

$$\Rightarrow T = -\frac{\mu}{k}\left(\frac{g_x}{\nu}\right)^2 \left[\frac{(h-y)^4}{12} - \frac{h^4}{12}\right] + T_0$$

$$T(h) = T_0 + \frac{\mu}{k}\left(\frac{g_x h^2}{\nu}\right)^2 \frac{1}{12}$$

$$= T_0 + \frac{c_p \mu}{k} \frac{(g_x h^2/\nu)^2}{12 c_p} = T_0 + \frac{Pr}{12} \frac{(g_x h^2/\nu)^2}{c_p}$$