Solutions to exam in Fluid Mechanics SG2214
2008-01-14

1.

x-momentum: $0 = -\underbrace{\frac{\partial p}{\partial x}}_{=0 \text{ since } p_{air} = \text{const.}} + \varrho g \cos\theta + \mu \frac{\partial^2 u}{\partial y^2}$

$\Rightarrow \tau_{xy} = \mu \frac{\partial u}{\partial y} = -\varrho g \cos\theta\, y + C$

$\tau_{xy}(y=h) = -\varrho g \cos\theta\, h + C = -\tau_{air}$

$\Rightarrow \tau_{xy} = -\tau_{air} + \varrho g \cos\theta (h-y)$

$\Rightarrow \mu u(y) = -\tau_{air} y + \varrho g \cos\theta \left(hy - \frac{y^2}{2}\right) + D$

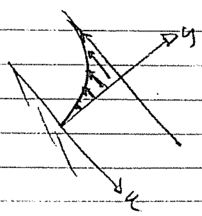
no slip $\Rightarrow u(0) = 0 \Rightarrow D = 0$

$$\boxed{u(y) = -\frac{\tau_{air}}{\mu} y + \frac{\varrho g \cos\theta}{\mu}\left(hy - \frac{y^2}{2}\right)}$$

$\tau_{xy}(y=0) = -\tau_{air} + \varrho g \cos\theta\, h = 0$

$\Rightarrow \tau_{air} = \varrho g \cos\theta\, h$

$\Rightarrow u(y) = -\frac{\varrho g \cos\theta}{\mu} \frac{y^2}{2}$



2. $F(z) = \left(\frac{m - i\Gamma}{2\pi}\right) \ln z \quad ; \quad z = re^{i\theta}$

a) $F = \left(\frac{m-i\Gamma}{2\pi}\right) \ln\left(re^{i\theta}\right) = \left(\frac{m-i\Gamma}{2\pi}\right)(\ln r + i\theta) =$

$= \frac{m \ln r}{2\pi} + \frac{\Gamma\theta}{2\pi} + i\left\{-\frac{\Gamma \ln r}{2\pi} + \frac{m\theta}{2\pi}\right\} =$

$= \Phi + i\psi$

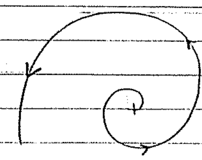
$\Rightarrow$ stream function $\psi = \frac{1}{2\pi}\left\{m\theta - \Gamma \ln r\right\}$

streamlines: $\psi = \text{const.}$

$\Rightarrow m\theta - \Gamma \ln r = m\theta_0 - \Gamma \ln r_0 = \text{const.}$

$\Rightarrow \theta - \theta_0 = \frac{\Gamma}{m} \ln(r/r_0)$

or $\boxed{\frac{r}{r_0} = e^{\frac{m}{\Gamma}(\theta - \theta_0)}}$ where r_0, θ_0 is a point on the streamline



b) $F'(z) = \frac{m - i\Gamma}{2\pi} \frac{1}{z} = \left(\frac{m-i\Gamma}{2\pi}\right) \frac{e^{-i\theta}}{r} =$

$= \left(\frac{m}{2\pi r} - i\frac{\Gamma}{2\pi r}\right) e^{-i\theta} \equiv (u_r - i u_\theta) e^{-i\theta}$

$\Rightarrow u_r = \frac{dr}{dt} = \frac{m}{2\pi r} \quad ; \quad u_\theta = \frac{r\,d\theta}{dt} = \frac{\Gamma}{2\pi r}$

$\frac{r\,dr}{dt} = \frac{m}{2\pi} \Rightarrow \frac{d}{dt}\left(\frac{r^2}{2}\right) = \frac{m}{2\pi}$

$\Rightarrow \frac{r^2}{2} - \frac{r_0^2}{2} = \frac{m}{2\pi}(t - t_0)$

$\Rightarrow r(t) = \sqrt{r_0^2 + 2\frac{m}{2\pi}(t-t_0)}$

From a) we have $\theta - \theta_0 = \frac{\Gamma}{2m} \ln(r/r_0)^2$

$\Rightarrow \theta(t) = \theta_0 + \frac{\Gamma}{2m} \ln\left(1 + \frac{m}{\pi r_0^2}(t - t_0)\right)$

3.

a) $\frac{\partial u}{\partial t} > 0 \;;\; u\frac{\partial u}{\partial x} = 0 \;;\; v\frac{\partial u}{\partial y} = 0 \;;$

$\omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{\partial u}{\partial y} > 0$

$\Gamma = U \cdot L$

b) $x > 0: \; \frac{\partial u}{\partial t} = 0 \;;\; u\frac{\partial u}{\partial x} > 0 \;;\; v\frac{\partial u}{\partial y} = 0 \;;\; \omega_z = 0$

$x < 0: \; \frac{\partial u}{\partial t} = 0 \;;\; u\frac{\partial u}{\partial x} < 0 \;;\; v\frac{\partial u}{\partial y} = 0 \;;\; \omega_z = 0$

$\Gamma = 0$ (since flow is irrotational)

c) $y > 0: \; \frac{\partial u}{\partial t} = 0 \;;\; u\frac{\partial u}{\partial x} = 0 \;;\; v\frac{\partial u}{\partial y} = 0 \;;\; \omega_z > 0$

$x < 0: \; \frac{\partial u}{\partial t} = 0 \;;\; u\frac{\partial u}{\partial x} = 0 \;;\; v\frac{\partial u}{\partial y} = 0 \;;\; \omega_z < 0$

$\Gamma = U_{max} \cdot L$

4.

2D flow $\underset{\sim}{\omega} = \omega_z(x,y)\, \underset{\sim}{e_z}$ $\qquad \underset{\sim}{u} = u(x,y)\underset{\sim}{e_x} + v(x,y)\underset{\sim}{e_y}$

$$\underset{\sim}{\omega}\cdot\nabla\underset{\sim}{u} = \omega_z \underbrace{\frac{\partial u}{\partial z}}_{=0}\underset{\sim}{e_x} + \omega_z \underbrace{\frac{\partial v}{\partial z}}_{=0}\underset{\sim}{e_y} = 0$$

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}0 - \frac{\partial u}{\partial y} = \omega_z(y)$$

$$\frac{D\underset{\sim}{\omega}}{Dt} = \underset{\sim}{e_z}\frac{D\omega_z}{Dt} = \underset{\sim}{e_z}\Big(\underbrace{\frac{\partial \omega_z}{\partial t}}_{=0} + \underbrace{u\frac{\partial\omega_z}{\partial x}}_{=0} + \underbrace{v\frac{\partial\omega_z}{\partial y}}_{=0}\Big) = 0$$

steady state

$$\nabla\rho = -\rho_0\alpha_0\nabla T = -\rho_0\alpha_0\Delta T_0\nabla\left(\frac{y}{h}\right) = -\rho_0\alpha_0\frac{\Delta T_0}{h}\underset{\sim}{e_y}$$

$$\nabla p = -\rho_0 g\nabla x = -\rho_0 g\underset{\sim}{e_x}$$

$$\frac{1}{\rho_0^2}\nabla\rho\times\nabla p = \frac{\alpha_0\Delta T_0 g}{h}\underset{\sim}{e_y}\times\underset{\sim}{e_x} = -\frac{\alpha_0\Delta T_0 g}{h}\underset{\sim}{e_z}$$

$$\frac{\mu_0}{\rho_0}\nabla^2\underset{\sim}{\omega} = \frac{\mu_0}{\rho_0}\nabla^2\omega_z(y)\underset{\sim}{e_z} = \frac{\mu_0}{\rho_0}\frac{\partial^2}{\partial y^2}\omega_z(y)\underset{\sim}{e_z}$$

Equation $\Rightarrow$ $0 = -\frac{\alpha_0\Delta T_0 g}{h} + \frac{\mu_0}{\rho_0}\frac{d^2\omega_z}{dy^2}$

$$\omega_z = -\frac{\partial u}{\partial y} = -\frac{U_0}{6h}\left(1 - 3\left(\frac{y}{h}\right)^2\right)$$

$$\Rightarrow \frac{d^2\omega_z}{dy^2} = \frac{U_0}{6h}\cdot\frac{6}{h^2} = \frac{U_0}{h^3}$$

$$\Rightarrow 0 = -\frac{\alpha_0\Delta T_0 g}{h} + \frac{\mu_0}{\rho_0}\frac{U_0}{h^3}$$

$\Rightarrow$ $\boxed{U_0 = \frac{\rho_0\alpha_0\Delta T_0 g h^2}{\mu_0}}$ $\Rightarrow$ equation satisfied

<u>Flow rate</u> $Q = \int_0^h u\,dy = \frac{U_0}{6}\int_0^h \frac{y}{h}\left(1 - \left(\frac{y}{h}\right)^2\right)dy =$

$$= \frac{U_0 h}{6}\int_0^1 y'(1-y'^2)\,dy' = \frac{U_0 h}{6}\left[\frac{y'^2}{2} - \frac{y'^4}{4}\right]_0^1 =$$

$$= \frac{U_0 h}{24}$$

5.

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) \quad ; \quad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$U = U_\infty - U_1(x,y)$ $\qquad U_1 \ll U_\infty$

$$\Rightarrow (U_\infty - U_1)\frac{\partial}{\partial x}(U_\infty - U_1) + V\frac{\partial}{\partial y}(U_\infty - U_1) = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)$$

$$\Rightarrow -U_\infty\frac{\partial U_1}{\partial x} - V\frac{\partial U_1}{\partial y} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)$$

Since $\frac{\partial}{\partial x}(U_\infty - U_1) + \frac{\partial V}{\partial y} = 0$

$$V \sim U_1\frac{\delta}{L} \ll U_\infty\frac{\delta}{L}$$

$$\underbrace{-U_\infty\frac{\partial U_1}{\partial x}}\ \underbrace{-V\frac{\partial U_1}{\partial y}} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)$$

$$\sim \frac{U_\infty U_1}{L} \gg \frac{U_1\delta}{L}\frac{U_1}{\delta}$$

$\Rightarrow$ linearized equation $-U_\infty\frac{\partial U_1}{\partial x} = \frac{\partial}{\partial y}\left(-\overline{u'v'}\right)$

with $-\overline{u'v'} = \beta_T U_\infty\theta_w\frac{\partial}{\partial y}(U_\infty - U_1)$

$$\Rightarrow \boxed{U_\infty\frac{\partial U_1}{\partial x} = \beta_T U_\infty\theta_w\frac{\partial^2 U_1}{\partial y^2}}$$

The total force, F_x, on the body is given by

$$F_x = \rho U_\infty^2\int_{-\infty}^{\infty}\frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy = \rho U_\infty^2\theta_w,$$

thus θ_w must be constant.

Thus, we have a constant turbulent diffusivity $\nu_T = \beta_T U_\infty\theta_w$ and

$$\left.\begin{aligned} &U_\infty\frac{\partial U_1}{\partial x} = \nu_T\frac{\partial^2 U_1}{\partial y^2} \\ &U_1 \to 0 \text{ as } y \to \infty \\ &\frac{\partial U_1}{\partial y} = 0 \text{ at } y = 0\end{aligned}\right\}$$

May be solved in analogy with laminar wake.

<u>See Recitation notes</u>

$$\Rightarrow \boxed{\frac{U_1}{U_\infty} = \sqrt{\frac{\theta_w}{4\pi\beta_T x}}\, e^{-\frac{y^2}{4\beta_T\theta_w x}}}$$