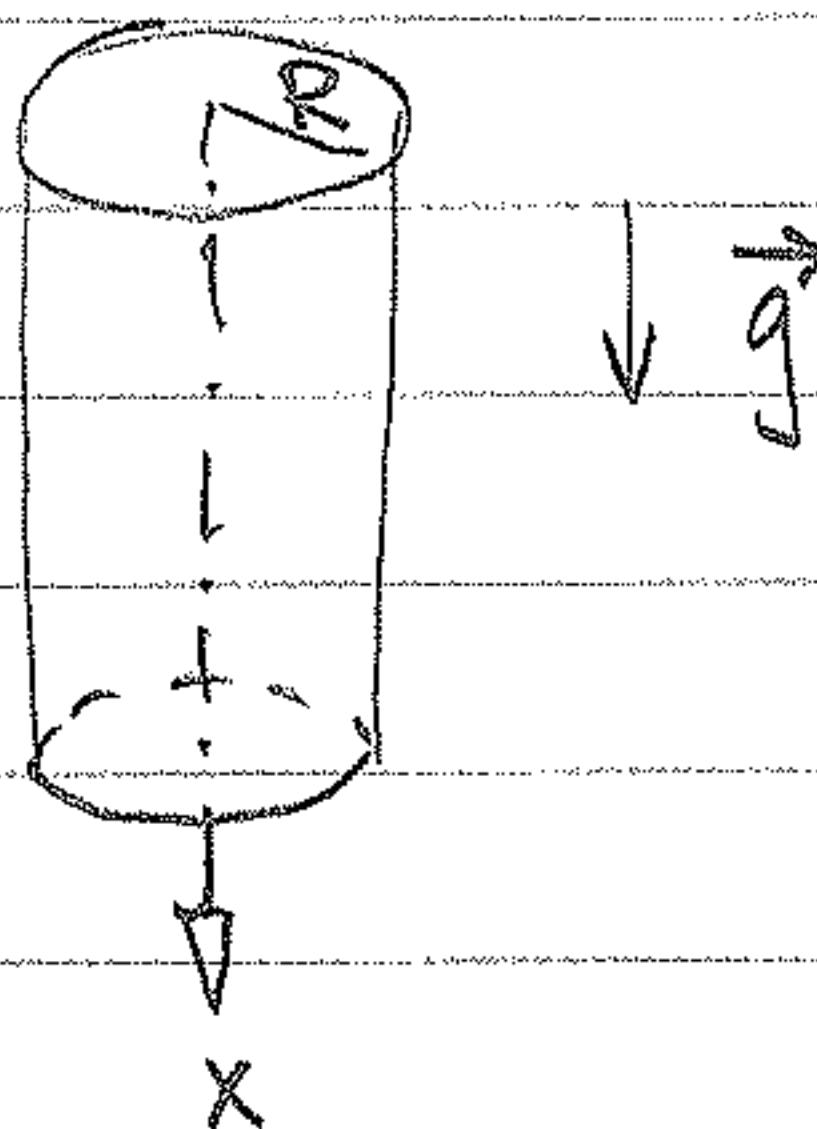


Solutions to exam in SG2214, 2011-01-11

1



Use cylindrical polar coordinates.

Fully developed flow
 $\Rightarrow \vec{u} = u(r) \hat{e}_x$

$$\text{Navier-Stokes' eq.'s} \Rightarrow 0 = g + V \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right)$$

$$\text{B.C. } u(r=R) = 0, u(r=0) < \infty$$

$$\text{Integrate} \Rightarrow r \frac{\partial u}{\partial r} = - \frac{gr^2}{2V} + A$$

$$\Rightarrow \frac{\partial u}{\partial r} = - \frac{gr}{2V} + \frac{A}{r} \Rightarrow u(r) = - \frac{gr^2}{4V} + A \ln r + B$$

$$u(r=R) = 0 \Rightarrow B = \frac{gR^2}{4V}; u \text{ finite at } r=0 \Rightarrow A=0$$

$$\therefore u(r) = \frac{gR^2}{4V} \left(1 - \left(\frac{r}{R} \right)^2 \right)$$

a)

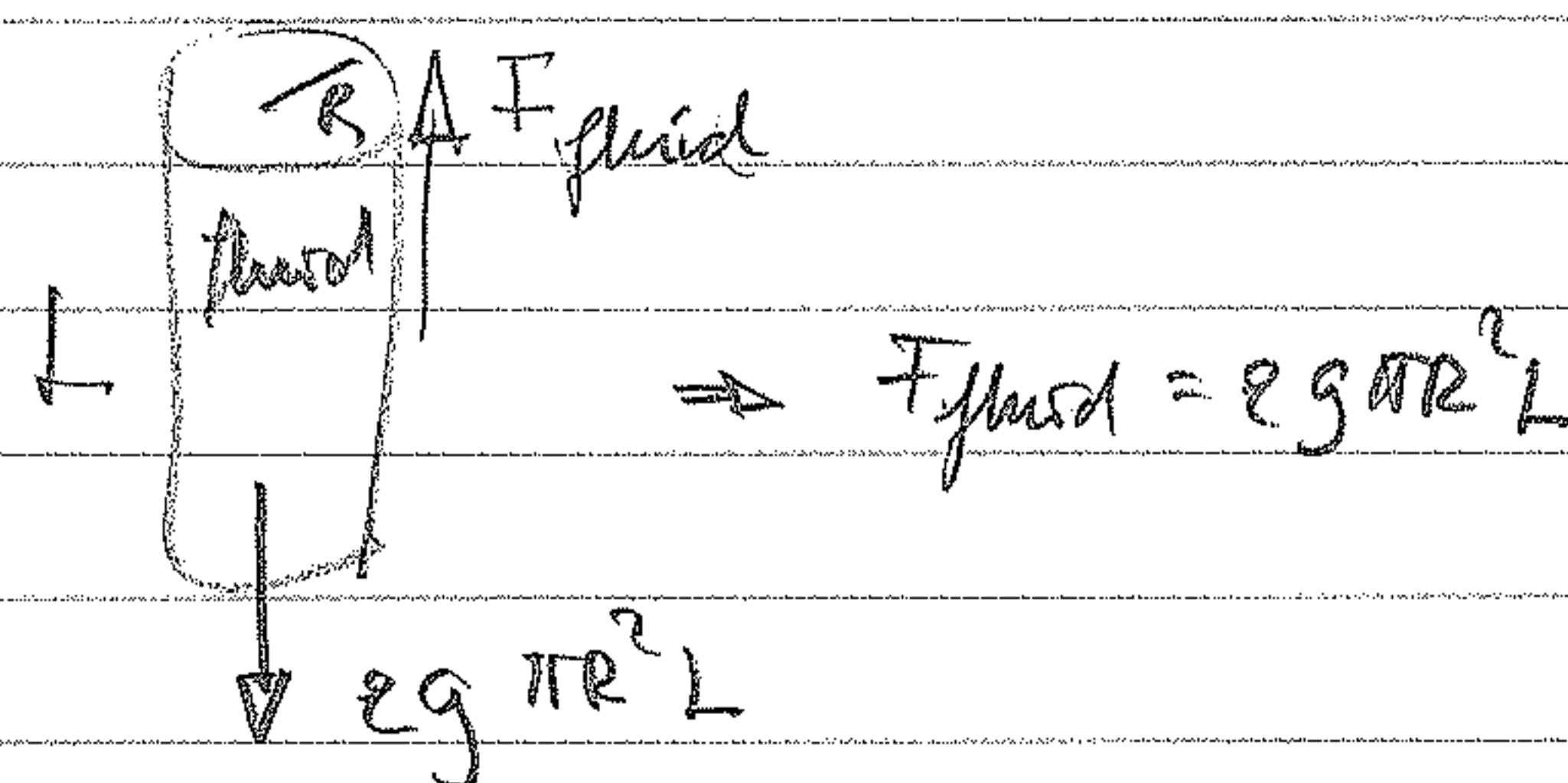
$$Q = \int_0^R u(r) 2\pi r dr = \frac{gR^2}{4V} 2\pi R \int_0^R \left(\frac{r}{R} - \left(\frac{r}{R} \right)^3 \right) dr =$$

$$= \frac{gR^2}{2V} \pi R^2 \int_0^1 (r^1 - r^{1^3}) dr^1 = \frac{gR^2}{2V} \pi R^2 \left(\frac{1}{2} - \frac{1}{4} \right) =$$

$$= \frac{gR^2}{8V} \pi R^2 \Rightarrow \bar{U} = \frac{Q}{\pi R^2} = \frac{gR^2}{8V}$$

- b) Force on the tube from the fluid is as large as the force on the fluid from the tube wall, but in the opposite direction.

Since the flow is fully developed the net flow of mass entering through a control volume is zero. Thus, the force on the fluid must cancel \Rightarrow

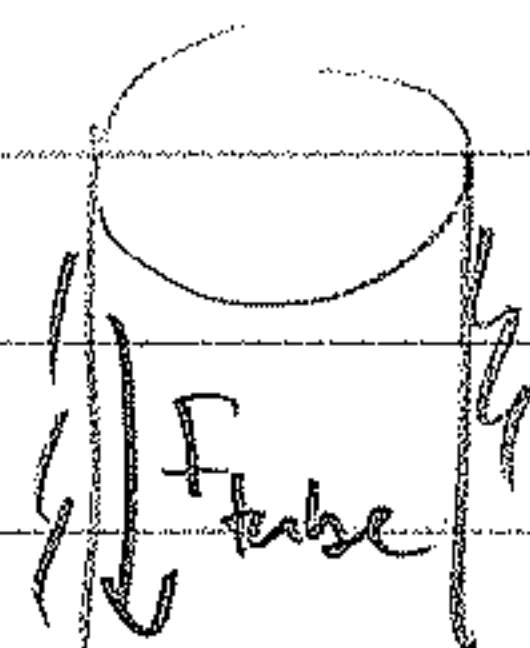


Thus the force on the tube is downwards

$$F_{\text{tube}} = \rho g \pi R^2 L$$

and we must
length

$$F_{\text{tube}} = \rho g \pi R^2 \text{ downwards}$$



(2)

$$a) (\nabla \times \vec{u})_2 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial x} = \frac{\omega k(y+H)}{kH} 2 \left(\sin(kx-\omega t) \right) = \frac{\omega k^2(y+H) \cos(kx-\omega t)}{kH}$$

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{\omega}{kH} \frac{\partial}{\partial y} \left(1 + \frac{1}{2} k^2 (y+H)^2 \right) \cos(kx-\omega t) = \\ &= \frac{\omega}{kH} \frac{1}{2} k^2 2(y+H) \cos(kx-\omega t) = \frac{\partial v}{\partial x} \end{aligned}$$

$$\Rightarrow (\nabla \times \vec{u})_2 = 0$$

$$b) \text{ Incompressible if } \nabla \cdot \vec{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{kH} \left(1 + \frac{1}{2} k^2 (y+H)^2 \right) (-k) \sin(kx-\omega t) =$$

$$= \frac{\omega}{kH} \left(1 + \frac{1}{2} (kH)^2 \left(\frac{y+H}{H} \right)^2 \right) (-k) \sin(kx-\omega t) \approx$$

$$\approx \frac{\omega}{kH} (-k) \sin(kx-\omega t)$$

$$\frac{\partial v}{\partial y} = \frac{\omega}{kH} k \sin(kx-\omega t) \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \approx 0$$

② At the bottom $y = -H$

$$\frac{\partial u}{\partial t} = \frac{aw^2}{kH} \sin(kx - \omega t)$$

$$u \frac{\partial u}{\partial x} = - \left(\frac{aw}{kH} \right)^2 k \cos(kx - \omega t) \sin(kx - \omega t)$$

$$\underbrace{v \frac{\partial u}{\partial y}}_{=0} = 0$$

$$a_x = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}$$

$$= \frac{aw^2}{kH} \sin(kx - \omega t) - \left(\frac{aw}{kH} \right)^2 k \cos(kx - \omega t) \sin(kx - \omega t)$$

$$a_y = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = 0$$

$$\text{since } v(y = -H) = 0$$

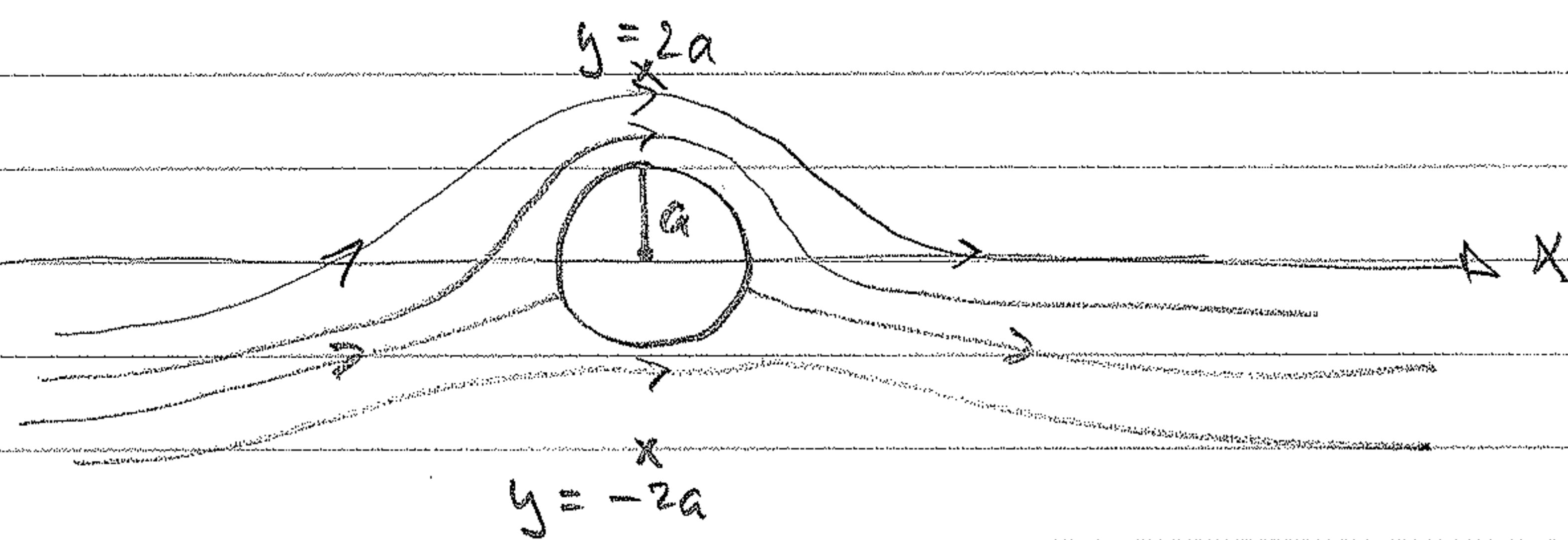
$$\text{Thus } \vec{a} = (a_x, 0)$$

③ a) $F = U_\infty \left(z + \frac{a^2}{z} \right) + \frac{i\Gamma}{2\pi} \ln z/a$

$$\Psi = \ln |F| = U_\infty \sin \theta \left(r - \frac{a^2}{r} \right) + \frac{\Gamma}{2\pi} \ln r/a$$

free stream
+ dipole

line vortex

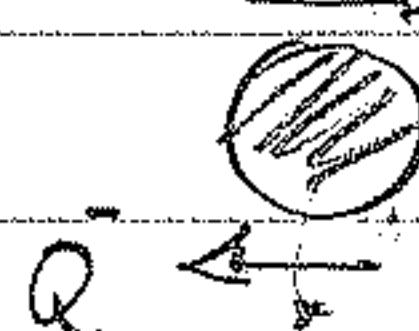


The free stream + dipole is a symmetric flow and gives no contribution to the diff. of fluxes above and below the cylinder.

The line vortex gives equal fluxes, but of opposite directions above and below the cylinder.

Above cylinder $Q_p^+ = \frac{\Gamma}{2\pi} \left(\ln \frac{2a}{a} - \ln \frac{a}{a} \right) = \frac{\Gamma \ln 2}{2\pi}$

below cylinder $Q_p^- = \frac{\Gamma}{2\pi} \left(\ln \frac{a}{a} - \ln \frac{2a}{a} \right) = -\frac{\Gamma \ln 2}{2\pi}$



Thus $\Delta Q = \frac{\Gamma \ln 2}{\pi}$

$$\textcircled{3} \quad b) \quad F(z) = V_0 \left(1 - \frac{a^2}{z^2}\right) + i \frac{\Gamma}{2\pi z} = (u_r - iu_\theta) e^{-iz}$$

$$z = r e^{i\theta} \Rightarrow \dots \Rightarrow \begin{cases} u_r = V_0 \left(1 - \frac{a^2}{r^2}\right) \cos \theta \\ u_\theta = -V_0 \left(1 + \frac{a^2}{r^2}\right) \sin \theta - \frac{\Gamma}{2\pi r} \end{cases}$$

At $x=0, y=\pm a$ we have $r=a, \theta=\pm \pi/2$

$$\Rightarrow u_\theta^+ = \left[-V_0 \left(1 + \frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi r} \right]_{r=a} = -2V_0 - \frac{\Gamma}{2\pi a}$$

$$u_\theta^- = \left[V_0 \left(1 + \frac{a^2}{r^2}\right) - \frac{\Gamma}{2\pi r} \right]_{r=a} = 2V_0 - \frac{\Gamma}{2\pi a}$$

and $u_r = 0$.

$$\text{Bernoulli} \Rightarrow p + \frac{1}{2} \rho u_z^2 = p_a + \frac{1}{2} \rho U_0^2$$

$$p_- - p_+ = \frac{1}{2} \rho (u_\theta^+ - u_\theta^-) =$$

$$= \frac{1}{2} \rho \left(\frac{4U_0^2 + (\Gamma)^2 + 4U_0\Gamma}{2\pi a} - \frac{4U_0^2 + (\Gamma)^2 - 4U_0\Gamma}{2\pi a} \right)$$

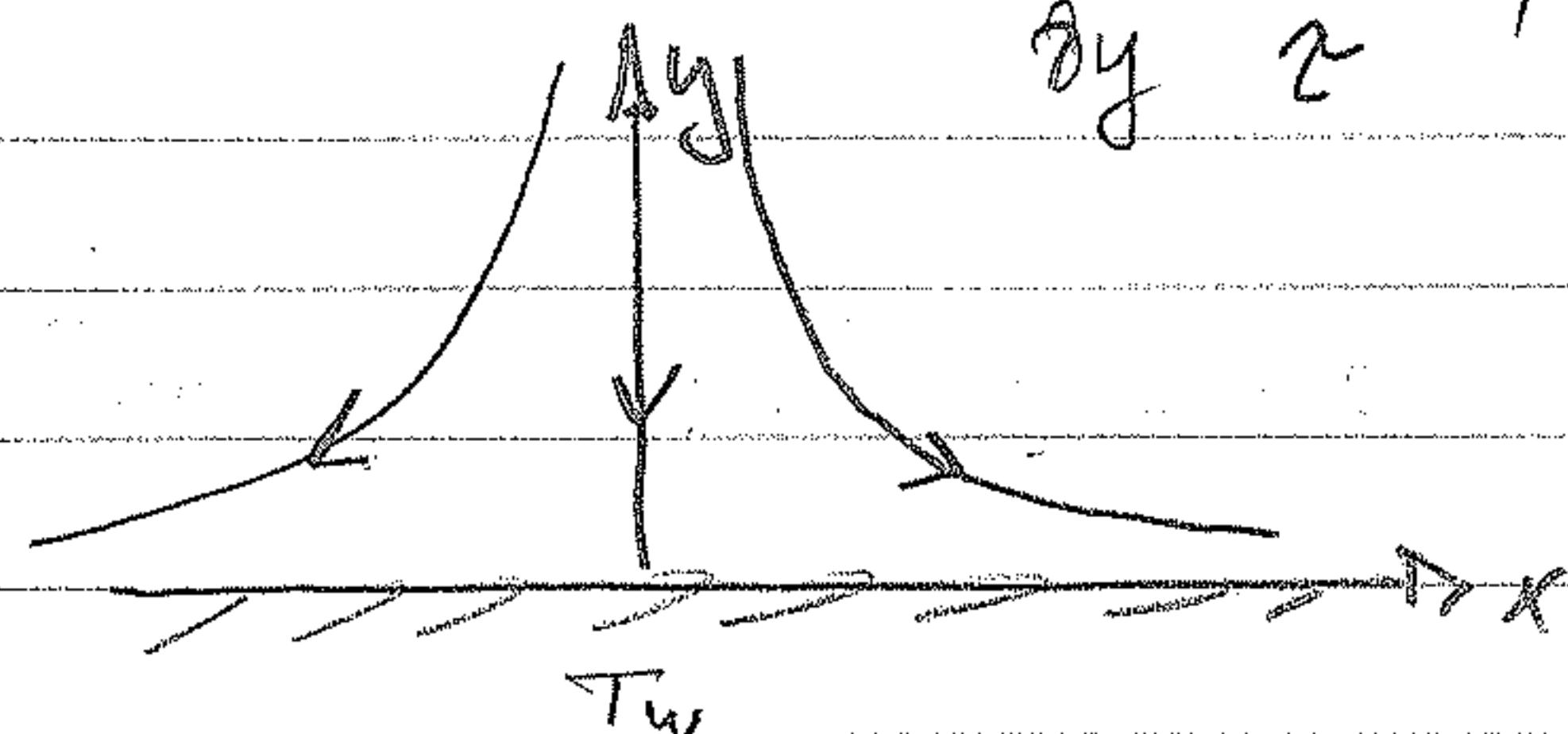
$$= \frac{1}{2} \rho \frac{4U_0\Gamma}{\pi a}, \quad \frac{2 \rho U_0 \Gamma}{\pi a}$$

(4)

See lecture notes or course book.

(5)

a) $\Psi = xy/2 \Rightarrow u = \frac{\partial \Psi}{\partial y} = x, v = -\frac{\partial \Psi}{\partial x} = -\frac{y}{2}$



Energy eq., steady state, dissipation neglected
and $T = T(y)$

$$\Rightarrow \underbrace{\frac{u \frac{dT}{dx} + v \frac{dT}{dy}}{\rho c}}_{=0} = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + 0 \quad \text{diss. neglected}$$

a)

$$\Rightarrow \int -\frac{y}{c} \frac{dT}{dy} = k \frac{d^2 T}{dy^2}$$

$$T(y=0) = T_w$$

$$T(y \rightarrow \infty) \rightarrow T_\infty$$

$$\Leftrightarrow T'' + \frac{y}{k} T' = 0 \Rightarrow T' = A e^{-y^2/(2k)} \quad \text{IC: } y=0, T'=0$$

$$\Rightarrow T(y) = A \int_0^y e^{-y'^2/(2k)} dy' + B$$

$$T(0) = T_w \Rightarrow B = T_w$$

⑤ a) coat.

$$T(\alpha) = T_{\infty} \Rightarrow T_{\alpha} = A \int_0^{\infty} e^{-y^2/(2k\tau)} dy' + T_w$$

$$T_{\alpha} = A \sqrt{2k\tau} \int_0^{\infty} e^{-\left(\frac{y'}{\sqrt{2k\tau}}\right)^2} \frac{dy'}{\sqrt{2k\tau}} + T_w$$

$$= \sqrt{\pi} \text{ according to hint}$$

$$\Rightarrow A \sqrt{2k\tau} = T_{\alpha} - T_w$$

$$\sqrt{\pi}/2$$

$$\therefore T(y) = T_w - \frac{(T_w - T_{\alpha})}{\sqrt{2k\tau}} \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2/(2k\tau)} dz'$$

$$= T_w - (T_w - T_{\alpha}) \frac{2}{\sqrt{\pi}} \int_0^{y/\sqrt{2k\tau}} e^{-z^2} dz'$$

$$= \operatorname{erf}(y/\sqrt{2k\tau})$$

b) Heat flux density $\vec{q} = -k \nabla T = -k \frac{dT}{dy} \vec{e}_y$

$$q_y = -k \frac{dT}{dy} = k (T_w - T_{\alpha}) \frac{2}{\sqrt{2k\tau}} e^{-y^2/2k\tau}$$

At the wall $y=0$: $q_y(y=0) = k (T_w - T_{\alpha}) \frac{2}{\sqrt{2k\tau}} \frac{1}{\sqrt{\pi}}$