

## 1. Momentum equation

$$\rho \left( \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = - \underbrace{\frac{\partial p}{\partial x_i}}_{\text{net surface force}} + \mu \underbrace{\frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\text{net viscous surface force per unit vol. of small fluid element}} + \underbrace{\rho f_i}_{\text{body force}}$$

acceleration of material fluid particle      net surface force from pressure per unit volume of small fluid element      net viscous surface force per unit vol. of small fluid element

Density  $\rho$ , Dynamic viscosity  $\mu$ , Pressure  $p$

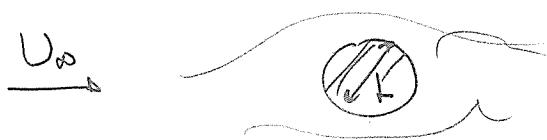
All terms have the unit of force/unit volume  
i.e. ( $N/m^3$ )

Continuity equation

$$\underbrace{\frac{\partial u_n}{\partial x_n}}_{\text{bulk strain rate}} = 0 \quad (1/s)$$

bulk strain rate

$$= \frac{1}{\Delta V} \frac{D}{Dt} \Delta V$$



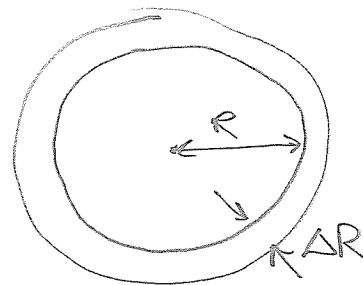
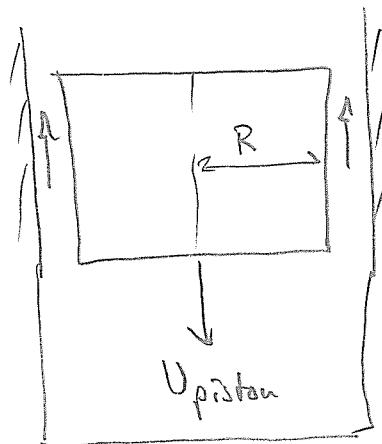
$$Re = \frac{\rho U_0 L}{\mu} \sim \frac{\text{inertial forces}}{\text{viscous forces}}$$

If  $Re \gg 1$  then  $\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$  is smaller.

$$\begin{aligned} \epsilon_{ijk} \frac{\partial w_k}{\partial x_i} &= \epsilon_{imn} \frac{\partial}{\partial x_i} \left( \epsilon_{kln} \frac{\partial u_m}{\partial x_l} \right) = \epsilon_{kij} \epsilon_{kln} \frac{\partial^2 u_m}{\partial x_j \partial x_e} = \\ &= (\delta_{ie} \delta_{jm} - \delta_{im} \delta_{je}) \frac{\partial^2 u_m}{\partial x_j \partial x_e} = \underbrace{\frac{\partial^2 u_j}{\partial x_g \partial x_i}}_{=0} - \underbrace{\frac{\partial^2 u_i}{\partial x_f \partial x_j}}_{=0} \end{aligned}$$

2.

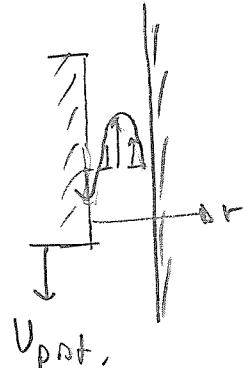
a)



$$Q_{\text{gap}} = \pi R^2 U_{\text{piston}} \text{ upwards}$$

- b) Force  $F$  must overcome pressure difference  $\Delta p$  between the piston ends.  $F = \Delta p \pi R^2$   
 One can also include friction force from fluid in the gap. Then

$$F = \Delta p \pi R^2 + \mu \frac{du}{dr} \Big|_{\text{piston}} \cdot 2\pi R \cdot L$$



Assuming a plane channel with  $r=0$  at piston wall.

$$u(r) = \bar{U} \frac{r}{\Delta R} \left(1 - \frac{r}{\Delta R}\right) - U_{\text{piston}} \left(1 - \frac{r}{\Delta R}\right)$$

$\bar{U}$  must be such that  $Q_{\text{gap}} = \pi R^2 U_{\text{piston}}$

$$\Rightarrow \bar{U} \Delta R \cdot 2\pi R - \frac{U_{\text{piston}} \Delta R \cdot 2\pi R}{2} = U_{\text{piston}} \pi R^2$$

$$\Rightarrow \bar{U} = \frac{U_{\text{piston}}}{2} \left( \frac{R}{\Delta R} + 1 \right)$$

The pressure difference is given by N.S. eq.

$$0 = -\frac{(-\Delta p)}{L} + \mu \frac{d^2 u}{dr^2} \Rightarrow \Delta p = 12\mu \frac{\bar{U}L}{(\Delta R)^2} = \\ = \frac{12\mu L}{(\Delta R)^2} \frac{U_{\text{piston}}}{2} \left( \frac{R}{\Delta R} + 1 \right)$$

The velocity gradient

$$\frac{du}{dr}_{\text{piston}} = \frac{6 \bar{U}}{\Delta R} + \frac{U_{\text{piston}}}{\Delta R} = \frac{U_{\text{piston}}}{\Delta R} \left( \frac{3R}{\Delta R} + 4 \right)$$

$$\Rightarrow F = 6\pi\mu L U_{\text{piston}} \left( \frac{R}{\Delta R} \right)^3 \left( 1 + \frac{\Delta R}{R} \right) + 6\pi\mu L U_{\text{piston}} \left( \frac{R}{\Delta R} \right)^2 \left( 1 + \frac{4}{3} \frac{\Delta R}{R} \right)$$

Thus, since  $\frac{\Delta R}{R} \ll 1$  we have

$$F \approx 6\pi\mu L U_{\text{piston}} \left( \frac{R}{\Delta R} \right)^3.$$

other terms are a factor  $\frac{\Delta R}{R} \ll 1$  smaller.

③ Let  $z = r e^{i\theta}$

$$\Rightarrow F = \frac{2}{3} U_0 L e^{-i\frac{\pi}{2}} \left( \frac{r e^{i\theta}}{L} \right)^{3/2} = \frac{2}{3} U_0 L \left( \frac{r}{L} \right)^{3/2} e^{i\left(\frac{3\theta}{2} - \frac{\pi}{2}\right)}$$

$$= \frac{2}{3} U_0 L \left( \frac{r}{L} \right)^{3/2} \left[ \cos\left(\frac{3\theta}{2} - \frac{\pi}{2}\right) + i \sin\left(\frac{3\theta}{2} - \frac{\pi}{2}\right) \right]$$

$$= \phi + i \psi$$

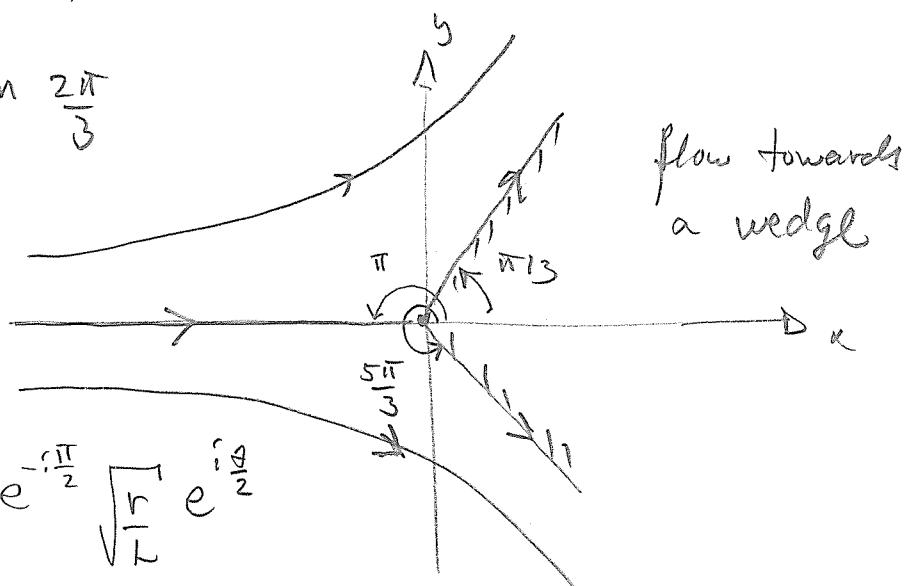
$\uparrow$  streamfunction

a)  $\Rightarrow \psi = \frac{2}{3} U_0 L \left( \frac{r}{L} \right)^{3/2} \sin\left(\frac{3\theta}{2} - \frac{\pi}{2}\right)$

Streamlines with  $\psi = 0 \Rightarrow \sin\left(\frac{3\theta}{2} - \frac{\pi}{2}\right) = 0$

$$\Rightarrow \frac{3\theta}{2} - \frac{\pi}{2} = n\pi, \quad n = 0, 1, 2, \dots$$

$$\Rightarrow \theta = \frac{\pi}{3} + n \frac{2\pi}{3}$$



$$u - i v = F^1 = U_0 e^{-i\frac{\pi}{2}} \sqrt{\frac{r}{L}} e^{i\frac{\theta}{2}}$$

stagnation point at  $r = 0$

$$u = U_0 \sqrt{\frac{r}{L}} \cos \frac{1}{2}(\theta - \pi), \quad v = -U_0 \sqrt{\frac{r}{L}} \sin \frac{1}{2}(\theta - \pi)$$

b) Bernoulli's eq.  $\Rightarrow p + \frac{1}{2} \rho (u^2 + v^2) = P_0$

$$u^2 + v^2 = U_0^2 \frac{r}{L} \cos^2 \frac{1}{2}(\theta - \alpha) + U_0^2 \frac{r}{L} \sin^2 \frac{1}{2}(\theta - \alpha) = \\ = U_0^2 \frac{r}{L}$$

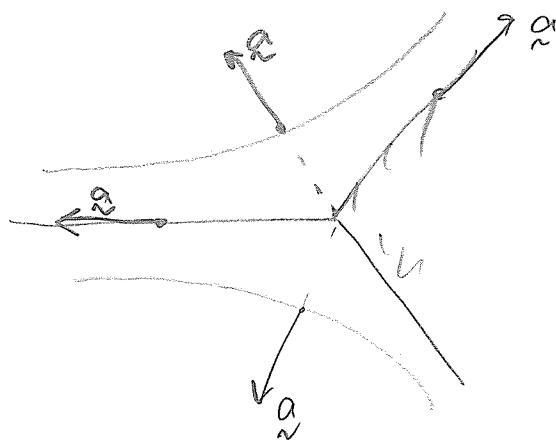
$$\Rightarrow p = P_0 - \frac{1}{2} \rho U_0^2 \frac{r}{L}$$

c) In Rotational flow (steady state)

$$\underline{\underline{u}} \cdot \nabla \underline{\underline{u}} = -\frac{1}{\rho} \nabla p = \rho U_0^2 \nabla \left( \frac{r}{L} \right) = \frac{\rho U_0^2}{L} \hat{e}_r$$

acceleration vector of fluid particle

Acceleration in radial direction



4.

a) Boundary layer exists if  $Re = \frac{U_\infty x}{\nu} \gg 1$

$$b) \underbrace{\epsilon c_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right)}_{\text{Rate of change of thermal energy per unit volume of material fluid element.}} = \underbrace{k \frac{\partial^2 T}{\partial y^2}}_{\text{Net rate of heat added to fluid element per unit volume due to heat conduction}} + \underbrace{\mu \left( \frac{\partial u}{\partial y} \right)^2}_{\text{Dissipation: deformation work rate of viscous forces per unit volume}}$$

Rate of change of thermal energy per unit volume of material fluid element.

Net rate of heat added to fluid element per unit volume due to heat conduction

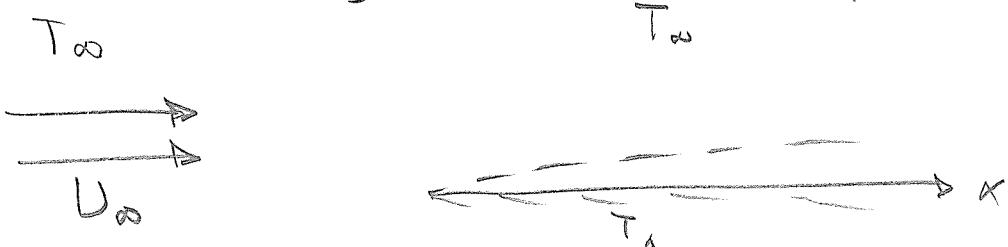
Dissipation: deformation work rate of viscous forces per unit volume

The complete dissipation function is  $\Phi = 2\mu \epsilon_{ij} e_{ij}$   
 $\left( \frac{\partial u}{\partial y} \right)^2$  is much larger than other terms in a b.l.

Also  $k \nabla^2 T = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$  is approximated

with  $k \frac{\partial^2 T}{\partial y^2}$  since  $\frac{\partial^2 T}{\partial x^2} \ll \frac{\partial^2 T}{\partial y^2}$  in b.l.

Boundary conditions on flat plate



$$T(x=0, y) = T_\infty, \quad T(x, y=0) = T_0, \quad T(x, y \rightarrow \infty) \rightarrow T_\infty$$

9 We need  $u$  &  $v$  from Blasius

$$u = \frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y} \left( U_\infty \sqrt{Vx/U_\infty} f(\eta) \right) = U_\infty \sqrt{\frac{Vx}{U_\infty}} f' \frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{Vx/U_\infty}} \Rightarrow u = U_\infty f'(\eta), \quad \eta = \frac{y}{\sqrt{Vx/U_\infty}}$$

$$v = -\frac{\partial \Psi}{\partial x} = -U_\infty \frac{1}{2} \frac{1}{\sqrt{Re_x}} f(\eta) - U_\infty \sqrt{\frac{Vx}{U_\infty}} f' \frac{\partial \eta}{\partial x}$$

$$= -U_\infty \frac{1}{2} \frac{1}{\sqrt{Re_x}} [f(\eta) - \eta f'], \quad Re_x = \frac{U_\infty x}{V}$$

$$\frac{\partial T}{\partial x} = T_\infty \theta'(\eta) \frac{\partial \eta}{\partial x} = -T_\infty \theta'(\eta) \frac{\eta}{2x}$$

$$\frac{\partial T}{\partial y} = T_\infty \theta'(\eta) \frac{\partial \eta}{\partial y} = T_\infty \theta'(\eta) \frac{1}{\sqrt{Vx/U_\infty}} = T_\infty \theta'(\eta) \frac{\sqrt{Re_x}}{x}$$

$$\frac{\partial u}{\partial y} = U_\infty f''(\eta) \frac{1}{\sqrt{Vx/U_\infty}}; \mu \left( \frac{\partial u}{\partial y} \right)^2 = U_\infty^2 (f')^2 \frac{\rho U_\infty}{x}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = -U_\infty f' T_\infty \frac{\theta' \eta}{2x} - \frac{U_\infty}{2x} T_\infty \theta' (f - \eta f')$$

$$= -\frac{U_\infty T_\infty}{2x} \theta' f$$

$$k \frac{\partial^2 T}{\partial y^2} = k T_\infty \theta'' \frac{U^2}{(Vx/U_\infty)} = \frac{\rho C_p}{Pr} T_\infty U_\infty \frac{\theta''}{x}; \quad Pr = \frac{\nu \rho C_p}{k}$$

B.D. eq.  $\Rightarrow$

$$-\frac{\rho C_p U_\infty T_\infty}{2} \frac{f \theta'}{x} = \frac{\rho C_p U_\infty T_\infty}{x} \frac{\theta''}{Pr} + \frac{\rho U_\infty^3}{x} (f'')^2$$

Divide with  $\frac{\rho C_p U_\infty T_\infty}{x} + \frac{1}{Pr}$   $\Rightarrow$

$$\theta'' + \frac{Pr f \theta'}{2} + \frac{U_\infty^2}{C_p T_\infty} Pr (f'')^2 = 0$$

d)  $Pr=1 \Rightarrow \theta'' + \frac{1}{2} f \theta' + \frac{U_\infty^2}{C_p T_\infty} (f'')^2 = 0,$

$$\theta = -\frac{U_\infty^2}{2 C_p T_\infty} (f')^2 + A f' + B$$

$$\theta' = -\frac{U_\infty^2}{C_p T_\infty} f' f'' + A f'''$$

$$\theta'' = -\frac{U_\infty^2}{C_p T_\infty} [(f'')^2 + f' f'''] + A f''''$$

$$\Rightarrow -\frac{U_\infty^2}{C_p T_\infty} (f'')^2 - \cancel{\frac{U_\infty^2}{C_p T_\infty} f' f''} + A f''' - \cancel{\frac{U_\infty^2}{C_p T_\infty} \frac{1}{2} f' f'' f'} + \frac{1}{2} A f f''' + \frac{U_\infty^2}{C_p T_\infty} (f'')^2 = 0$$

Eq. is satisfied )

$$B.C. \Rightarrow \theta(0) = \frac{T_0}{T_\infty} \Rightarrow B = \frac{T_0}{T_\infty}$$

$$\theta(\eta \rightarrow 0) \rightarrow 1 \Rightarrow -\frac{U_\infty^2}{2 C_p T_\infty} + A + \frac{T_0}{T_\infty} = 1$$

$$A = 1 - \frac{T_0}{T_\infty} + \frac{U_\infty^2}{2 C_p T_\infty}$$

Heat flux density at wall

$$q_b(y=0) = -k \frac{\partial T}{\partial y} = -k \frac{T_\infty \theta'(0)}{\sqrt{Vx/U_\infty}} = -\frac{k T_\infty U_\infty}{V \sqrt{Re_x}} \theta'(0) =$$

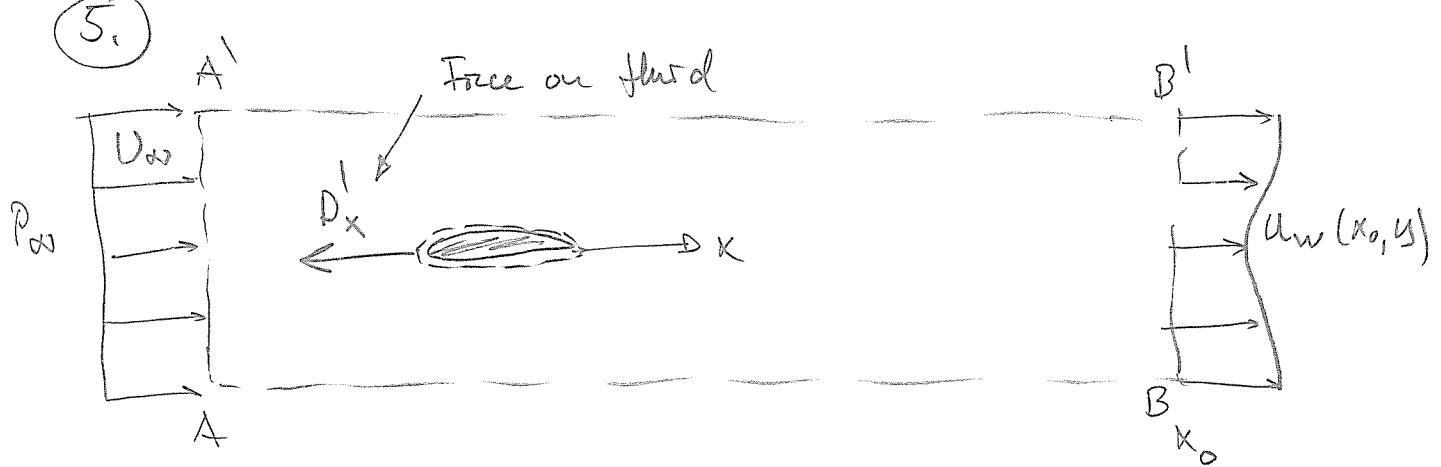
$$= -\underbrace{\frac{1}{Pr}}_{=1} \frac{\rho C_p U_\infty T_\infty}{\sqrt{Re_x}} \theta'(0)$$

$$\theta'(0) = A f''(0) = \left( 1 - \frac{T_0}{T_\infty} + \frac{U_\infty^2}{2C_p T_\infty} \right) f''(0)$$

$$q_b(y=0) = \frac{\rho C_p U_\infty}{\sqrt{Re_x}} f''(0) \left( T_0 - T_\infty - \frac{U_\infty^2}{2C_p} \right)$$

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(5.)



Steady state  $\frac{d}{dt} = 0$

$$\int_B^{B'} u_w dy - \int_A^{A'} U_\infty dy + 2 \int_{A'}^{B'} V_\infty(x) dx = 0$$

$$\Rightarrow \int_{-y_\infty}^{y_\infty} (U_\infty - u_w) dy = 2 \int_{A'}^{B'} V_\infty(x) dx$$

$$\int_B^{B'} 2 u_w^2 dy - \int_A^{A'} 2 U_\infty^2 dy + 2 \underbrace{\int_{A'}^{B'} 2 U_\infty V_\infty(x) dx}_{\epsilon U_\infty \int_{-y_\infty}^{y_\infty} (U_\infty - u_w) dy} = -D_x'$$

$$D_x' = \int_{-y_\infty}^{y_\infty} (\epsilon U_\infty^2 - \epsilon u_w^2 - 2 U_\infty^2 + \epsilon U_\infty u_w) dy =$$

$$= \int_{-y_\infty}^{y_\infty} (\epsilon U_\infty u_w - \epsilon u_w^2) dy = \epsilon U_\infty^2 \int_{-y_\infty}^{y_\infty} \frac{u_w}{U_\infty} \left(1 - \frac{u_w}{U_\infty}\right) dy$$

$$= \epsilon U_\infty^2 \int_{-\infty}^0 \frac{u_w}{U_\infty} \left(1 - \frac{u_w}{U_\infty}\right) dy$$