Homework problem 1(3p.), Fluid Mechanics SG2214
Due Sept 16, 2013

Be careful to explain and motivate each non-trivial step of the solution to these problems.

1. (1 p.) In the morning, at t=0 say, the temperature increases southwards in Sweden at a rate of $\Delta T/L$ [K/m]. A wind from the north blowing at U [m/s] brings cold air with it. However, radiation from the sun and ground heats the air, such that material air particles, going with the wind, increase their temperature with time at a given constant rate $\dot{T}$ [K/s].
Let $\dot{T} = 0.36$ K/hour, U=2 m/s and $\Delta T/L = 0.1$ K/km.

a) What is the time derivative of temperature for a stationary observer in [K/hour]?

b) In what direction, and at what speed should an observer move in order to experience a constant temperature?

c) Find an explicit expression for the temperature field in Eulerian coordinates with one space dimension if $T(t=0, x=0) = T_0 = 280$ K.

2. (1 p.) Use tensor notation to show that if $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ and $\nabla \cdot \mathbf{u} = 0$, then

a) $\nabla \cdot \boldsymbol{\omega} = 0$

b) $\nabla \times (\boldsymbol{\omega} \times \mathbf{u}) = \mathbf{u} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \mathbf{u}$

c) $\nabla \times \boldsymbol{\omega} = -\nabla^2 \mathbf{u}$

d) $\mathbf{u} \times \boldsymbol{\omega} = \nabla(\frac{\mathbf{u} \cdot \mathbf{u}}{2}) - \mathbf{u} \cdot \nabla \mathbf{u}$

3. (1 p.) Consider the two-dimensional flow field given by

$$u = a\omega \frac{\cosh\left[k\left(y+H\right)\right]}{\sinh\left[kH\right]} \sin\left(kx\right)\sin\left(\omega t\right) \; , \quad v = -a\omega \frac{\sinh\left[k\left(y+H\right)\right]}{\sinh\left[kH\right]} \cos\left(kx\right)\sin\left(\omega t\right)$$

The flow field is a linear model for a standing surface gravity wave on a water layer of depth H above a horizontal plane wall at $y= -H$, where a is the small constant amplitude of the wave surface at

$$y = \eta(x,t) \equiv a\cos(kx)\cos(\omega t).$$

u is the horizontal velocity component and v is the vertical component. Such a wave can be generated by the superposition of a travelling wave and its reflection on a vertical wall. ω is the frequency of the wave and $k=2\pi/\lambda$ is the wave number (λ is wavelength). A visualization of the fluid particle trajectories can be found at:
http://www.atmos.washington.edu/2006Q4/505/trajectories_standing_wave.jpg
There is also a movie on You Tube to watch:
http://www.youtube.com/watch?v=NpEevfOU4Z8

a) Sketch the surface of the wave at time *t=0*. What is the velocity field *t=0*? Calculate the acceleration of a fluid particle at the surface at *t=0*.

b) Make a sketch of the instantaneous streamlines at $\omega t=\pi/2$ that includes the origin and axes of the coordinate system. Calculate u and v for $kx = 0,\ \pi/2,\ \pi,\ 3\pi/2$ at $\omega t=\pi/2$ and indicate how this agrees qualitatively with the streamline pattern. Calculate the acceleration of a fluid particle at $kx=\pi/2$.

c) Show that the flow field is incompressible.

d) Calculate the velocity gradient tensor $\dfrac{\partial u_i}{\partial x_j}$.

e) Separate the flow field in its local **translation**, **rotation** and **deformation** for a fluid element at the bottom of the fluid layer at y= -H. (Note that the wave amplitude, a, is assumed much smaller than both the depth, H, and the wavelength, λ.) Illustrate this qualitatively for a small square aligned with the Cartesian coordinate axes at the phases $kx = 0,\ \pi/2,\ \pi,\ 3\pi/2$ and at $\omega t=\pi/2$. The relative motion may be written

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j\,,$$

where dx_j is the infinitesimal position vector relative to the point in question.

f) Assume the fluid is Newtonian with viscosity μ and calculate the stress vector $R_i(x,t)$ at a horizontal surface element next to the bottom surface at y=-H if the pressure field is given by

$$p = p_{atm} - \rho g y + \rho g a \frac{\cosh\left[k\left(y+H\right)\right]}{\cosh\left[kH\right]} \cos\left(kx\right)\cos\left(\omega t\right).$$

g) Does the flow field satisfy the boundary conditions of a Newtonian viscous fluid at the bottom wall?