Kinetic Theory of Gases

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1 Elements of Statistical Mechanics.

1.1 $6N$ dimensional phase space

From a mechanical point of law a gas is an assembly of a very large number N of molecules. To give the precise state of the gas at a moment we introduce its phase space. It is a space of $6N$ dimension, a huge space. A point in phase space is given by the total set of positions and velocities of all the molecules

$$x = (\mathbf{r}_1, \mathbf{c}_1, ..., \mathbf{r}_N, \mathbf{c}_N). \tag{1}$$

It is important to realize that the $\mathbf{r}_i$ and $\mathbf{c}_i$ are independent variables.

We can write the equations of motion as (V is the total potential energy of the interactions between the particles and possibly some external force)

$$\frac{d\mathbf{r}_i}{dt} = \mathbf{c}_i, \frac{d\mathbf{c}_i}{dt} = -\frac{1}{m}\frac{\partial V}{\partial \mathbf{r}_i}. \tag{2}$$

If we know the state of the gas at time t, we know a point in phase space. The equations of motion are first order in time, so there is a unique trajectory passing through the point in phase space.

1.2 Probability distribution in $6N$ dimensional phase space

But usually we have a very limited knowledge of the microscopic state of the gas. We can imagine a large number of copies of the gas, which all macroscopically have the same properties but different microscopic states. The number of copies within the volume element

$$d^{6N}x = d^3r_1 d^3c_1 ... d^3r_N d^3c_N \tag{3}$$

divided by the total number we write

$$\mathcal{F}d^{6N}x = \mathcal{F}d^3r_1 d^3c_1 ... d^3r_N d^3c_N. \tag{4}$$

$\mathcal{F}$ is thus a probability distribution in $6N$ dimensional phase space. As the total probability is 1 we must have

$$\int \mathcal{F} d^{6N}x = \int \mathcal{F} d^3r_1 d^3c_1...d^3r_N d^3c_N = 1 \qquad (5)$$

We know that the gas evolves in time. There is a flow in phase space. Its velocity is given by (2). We now see that this velocity field has a simple property, it has divergence zero in phase space. We recall that $\mathbf{r}_i$ and $\mathbf{c}_i$ are independent variables in phase space. We have

$$\frac{\partial}{\partial \mathbf{r}_1} \cdot \mathbf{c}_1 - \frac{\partial}{\partial \mathbf{c}_1} \cdot \frac{1}{m} \frac{\partial V}{\partial \mathbf{r}_1} + ... + \frac{\partial}{\partial \mathbf{r}_N} \cdot \mathbf{c}_N - \frac{\partial}{\partial \mathbf{c}_N} \cdot \frac{1}{m} \frac{\partial V}{\partial \mathbf{r}_N} = 0.$$

It has zero divergence. This means that it preserves volume in time,

$$d^{6N}x = d^3r_1 d^3c_1...d^3r_N d^3c_N.$$

When we follow the motion of a small volume in $6N$ dimensional phase space, the probability in that volume

$$\mathcal{F} d^{6N}x = \mathcal{F} d^3r_1 d^3c_1...d^3r_N d^3c_N$$

is also preserved. This means that the distribution function $\mathcal{F}$ is constant when we follow the motion in phase space.

1.3   Liouville's equation

Let us write this out explicitly

$$\frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial t} + \frac{dx}{dt} \cdot \frac{\partial \mathcal{F}}{\partial x} = 0 \qquad (6)$$

or

$$\frac{d\mathcal{F}}{dt} = \frac{\partial \mathcal{F}}{\partial t} + \sum_{i=1}^{N} (\mathbf{c}_i \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{r}_i} - \frac{1}{m} \frac{\partial V}{\partial \mathbf{r}_i} \cdot \frac{\partial \mathcal{F}}{\partial \mathbf{c}_i}) = 0. \qquad (7)$$

This is *the Liouville equation*. It is an exact consequence of the Newton equations of motion.

If we choose a distribution function as a δ function, so that at each time t every particle has well-defined position $\mathbf{r}_i(t)$ and velocity $\mathbf{c}_i(t)$

$$\mathcal{F} = \prod \delta(\mathbf{r}_i - \mathbf{r}_i(t))\delta(\mathbf{c}_i - \mathbf{c}_i(t)), \qquad (8)$$

the distribution function satisfies the Liouville equation if and only if $\mathbf{r}_i(t)$ and $\mathbf{c}_i(t)$ satisfy Newton's law. (2)

1.4 One particle distribution function

If we integrate out all variables except those belonging to the first particle we obtain

$$\int \mathcal{F} d^3 r_2 d^3 c_2 ... d^3 r_N d^3 c_N.$$

If we multiply this by $d^3 r_1 d^3 c_1$ we obtain the probability to find the first particle in $d^3 r_1 d^3 c_1$ irrespective of where all the other particles are. As the particles are identical, the expected number of particles in $d^3 r_1 d^3 c_1$ is $\mathcal{F}_1 d^3 r_1 d^3 c_1$ is

$$\mathcal{F}_1(\mathbf{r_1}, \mathbf{c_1}) = N \int \mathcal{F} d^3 r_2 d^3 c_2 ... d^3 r_N d^3 c_N.$$

This is called the reduced one-particle distribution function. Note that

$$\int \mathcal{F}_1 d^3 r_1 d^3 c_1 = N.$$

1.5 Equilibrium distribution, given energy

In equilibrium the distribution has to be time independent. But we know that the value of $\mathcal{F}$ is not changing as the point in phase space is moving. This means that for $\mathcal{F}$ to be time independent, it has to have the same value in all points it can reach in its motion. Suppose the system is isolated, so that its total energy is constant in time. The totalt energy is a macroscopic quantity. We assume that we know that it lies in a small intervall $E, E + \triangle E$. The assumption that the system comes arbitrarily close to any state with the same energy is called the *ergodic hypothesis*. If it holds it is clear that $\mathcal{F}$ simply has to be a constant.

So the equilibrium distribution function when the energy is given is simply a constant. But the total probability has to be 1, see (5). In other words, $\mathcal{F}$ is the volume of the energy shell to the power -1.

1.6 A dilute gas

Now we consider a not too dense gas. The interaction between the molecules then only act when the molecules are very close to each other. For hard spheres the interaction simply means that a small fraction of the volume is forbidden. When we calculate the total energy we can neglect the interactions. This means that the total energy is

$$E = \sum_{i=1}^{N} \frac{mc_i^2}{2} = \frac{m}{2}(c_{1x}^2 + c_{1y}^2 + c_{1z}^2 + ...c_{Nz}^2).$$

Or

$$c_{1x}^2 + c_{1y}^2 + c_{1z}^2 + ...c_{Nz}^2 = \frac{2E}{m}$$

We can think of $c_{1x}, c_{1y}, c_{1z}, c_{2x}, ...c_{Nz}$ as cartesian coordinates in a $3N$ dimensional space. It is clear that the distance to the origin is $R = \sqrt{2E/m}$.

We recall that $\mathcal{F}$ is simply a constant in the energy shell (and zero outside of it). But the total probability has to be 1, see (5). In other words, $\mathcal{F}$ is the volume of the energy shell to the power -1.

The volume of a sphere in velocity space with radius R is proportional to R^{3N} or to $E^{3N/2}$. The volume of a shell of thickness $\triangle E$ is thus proportional to $E^{3N/2-1}\triangle E$. The integration in space only gives V^N. Hence $\mathcal{F}$ is proportional to $E^{-3N/2+1}V^{-N}\triangle E$.

1.7 The one particle distribution, the Maxwellian

Can we find $\mathcal{F}_1$? One of the particle has a given velocity $\mathbf{c}_1$ and hence a given kinetic energy. So suppose it is in $d^3r_1 d^3c_1$. This means that the other particles have the energy $E' = E - mc_1^2/2$ and $3(N-1)$ dimensions to play with. They will be in a $3(N-1)$ dimensional spherical shell with radius R'

$$R'^2 = c_{2x}^2 + c_{2y}^2 + c_{2z}^2 + ...c_{Nz}^2 = \frac{2}{m}(E - \frac{mc_1^2}{2})$$

and the same thickness $\triangle E$. Its volume is proportional to $V^{N-1}E'^{3(N-1)/2-1}\triangle E$. We find that the probability is proportional to

$$V^{-1}(E - \frac{mc_1^2}{2})^{\frac{3(N-1)}{2}-1}E^{-\frac{3N}{2}+1}$$
$$= V^{-1}(1 - \frac{mc_1^2}{2E})^{\frac{3(N-1)}{2}-1}E.$$

But

$$(1 - \frac{mc_1^2}{2E})^{\frac{3(N-1)}{2}-1} = \exp[(\frac{3(N-1)}{2}-1)\ln(1 - \frac{mc_1^2}{2E})]$$

and the number of molecules N is very large and also the total energy E is proportional to the number of molecules, so that E/N is independent of N. We can expand the logarithm and keep only the first non-vanishing term. The next term will be of order $1/N$ and can be neglected. The result is proportional to

$$V^{-1}\exp(-\frac{3N}{2E}\frac{mc_1^2}{2}).$$

So that $\mathcal{F}_1$ is proportional to

$$\frac{N}{V}\exp(-\frac{3N}{2E}\frac{mc_1^2}{2}).$$

As $\mathcal{F}_1$ is normalized to N we find that

$$\frac{N}{V}(\frac{3N}{4E})^{3/2}\exp(-\frac{3N}{2E}\frac{mc_1^2}{2}).$$

is a normalization constant.

From $\mathcal{F}_1$ we can calculate the pressure to obtain

$$p = \frac{N}{V}\frac{2E}{3N}.$$

But for a not too dense gas, the gas law holds. This gives this gives us

$$\frac{2E}{3N} = kT,$$

so that $(n = N/V)$

$$\mathcal{F}_1 = n(\frac{m}{2\pi kT})^{3/2}\exp(-\frac{mc_1^2}{2kT}).$$

2 BBGKY Hierarchy

We have already introduced the one-particle distribution function. What happens if we integrate out all positions and momenta in the Liouville equation, will that result in an equation for the one particle distribution function? The potential energy is given by an exterior force and mutual forces.

$$V = \sum_{i=1}^{N} V(\mathbf{r}_i) + \sum_{1 \leq i < j \leq N} V_{12}(|\mathbf{r}_j - \mathbf{r}_j|)$$

If we cannot neglect the interaction between the molecules, the resulting equation also contains the reduced two particle distribution function,

$$\mathcal{F}_2 = N(N-1)\int F d^3r_3 d^3c_3...d^3r_N d^3c_N.$$

In fact, we obtain (here, $\mathbf{F} = -\boldsymbol{\nabla}V$)

$$\frac{\partial \mathcal{F}_1}{\partial t} + \mathbf{c}_1 \cdot \frac{\partial \mathcal{F}_1}{\partial \mathbf{r}_1} + \frac{\mathbf{F}}{m} \cdot \frac{\partial \mathcal{F}_1}{\partial \mathbf{c}_1} = -\int \{(\frac{\partial \mathcal{F}_2}{\partial \mathbf{c}_2} - \frac{\partial \mathcal{F}_2}{\partial \mathbf{c}_1}) \cdot \frac{1}{m}\frac{1}{r}\frac{\partial V_{12}}{\partial r}(\mathbf{r}_2 - \mathbf{r}_1)d^3r_2 d^3c_2 \quad (9)$$

The left hand side is a free streaming operator, the same as in the Boltzmann equation. But the right hand side involves the two particle distribution function.

2.1 Boltzmann equation as limit for a dilute gas

The parameter $\delta = n^{-1/3}$ defines an average distance between molecules. d is a diameter of a molecule. As we have already seen d/δ is an important parameter. In a dilute gas this parameter is small compared to 1.

$$\gamma = d^3 n = (d/\delta)^3$$

is a measure of the relative volume taken up by the molecules.

If the particles are uncorrelated, we would have

$$\mathcal{F}_2(\mathbf{r}_1, \mathbf{c}_1, \mathbf{r}_2, \mathbf{c}_2) = \mathcal{F}_1(\mathbf{r}_1, \mathbf{c}_1)\mathcal{F}_1(\mathbf{r}_2, \mathbf{c}_2).$$

But there is also a correlation, so that

$$\mathcal{F}_2(\mathbf{r}_1, \mathbf{c}_1, \mathbf{r}_2, \mathbf{c}_2) = \mathcal{F}_1(\mathbf{r}_1, \mathbf{c}_1)\mathcal{F}_1(\mathbf{r}_2, \mathbf{c}_2) + \mathcal{G}_2(\mathbf{r}_1, \mathbf{c}_1, \mathbf{r}_2, \mathbf{c}_2).$$

However, one can show that $\mathcal{G}_2$ is a term γ^2 whereas $\mathcal{F}_1$ is of order γ. It is possible to relate $\mathcal{G}_2$ to $\mathcal{F}_1$. This makes it possible to calculate the right hand side of (9) to lowest non-trivial order in γ. The result is that it is

$$\int (\mathcal{F}_1(\mathbf{c}^*)\mathcal{F}_1(\mathbf{c}_1^*) - \mathcal{F}_1(\mathbf{c})\mathcal{F}_1(\mathbf{c}_1))d\sigma(\mathbf{n})c_{rel}d^3c_1.$$

Here,

$$
\begin{aligned}
\mathbf{c}^* &= \mathbf{c} - \mathbf{n}\mathbf{n} \cdot (\mathbf{c} - \mathbf{c}_1), \\
\mathbf{c}_1^* &= \mathbf{c}_1 + \mathbf{n}\mathbf{n} \cdot (\mathbf{c} - \mathbf{c}_1).
\end{aligned}
\tag{10}
$$

In other words, the Boltzmann equation has been arrived as a the first non-trivial approximation for small d^3n. - In the sequel we shall simply write $\mathcal{F}$ for the one particle distribution function $\mathcal{F}_1$.

3   Boundary conditions

The boundary of the gas is usually a solid wall. The gas molecules will then collide with the molecules of the wall. This is a more complicated process than collisions between the gas molecules.

Diffuse reflexion and half Maxwellian   There are two simplified models

which are often used. The most important one is that of *diffuse reflexion.* The wall is assumed to be in thermal equilibrium. The relaxation towards equilibrium in a solid body is usually very much faster than in a gas. The gas molecules are asssumed to come into thermal equilibrium with the wall and to be reemitted as a Maxwellian.

Let us call the normal pointing into the gas $\mathbf{n}$. After collision with the wall the molecules are given by

$$\mathcal{F}_{out} = n_w \beta_w^3 \pi^{-3/2} \exp(-\beta_w^2 c^2), \ \mathbf{c} \cdot \mathbf{n} > 0. \tag{11}$$

Here, T_w, is the temperature of the wall and $\beta_w^2 = m/2kT_w$. The parameter n_w is determined from the condition that the number of gas molecules is conserved.

To calculate the outgoing flux of molecules, we choose a y-axis in the direction of $\mathbf{n}$. Then

$$\int_{\mathbf{c}\cdot\mathbf{n}>0} \mathcal{F}_{out} c_y d^3 c \tag{12}$$

$$= n_w \beta_w^{-1} \pi^{-3/2} \int \exp(-\xi^2) \xi_y d^3 \xi$$

$$= n_w \beta_w^{-1} \pi^{-3/2} \left(\int_{-\infty}^{\infty} \exp(-\xi_x^2) d\xi_x\right)^2 \left(\int_0^{\infty} \exp(-\xi_y^2) \xi_y d\xi_y\right)$$

$$= n_w \beta_w^{-1} \pi^{-3/2} (\pi^{1/2})^2 \frac{1}{2} = n_w \sqrt{\frac{kT_w}{2\pi m}}.$$

The factor $\sqrt{kT_w/2\pi m}$ is of the order of the thermal speed of gas molecules with the temperature T_w. When we know the incoming flux of gas molecules this gives the value of n_w.

Specular reflexion Another simplified model of collisions with the wall is that of *specular reflexion*. The molecules simply collide elastically with the wall. This means that

$$\mathcal{F}_{out}(\mathbf{c}) = \mathcal{F}_{in}(\mathbf{c} - 2\mathbf{n}(\mathbf{n}\cdot\mathbf{c})), \ \mathbf{c}\cdot\mathbf{n} > 0. \tag{13}$$

The molecules exchange momentum in the normal direction with the wall. They exert pressure on the wall. But no momentum in the tangential direction is exchanged. Hence, no shear stress on the wall. Further, there is no energy exhange with the wall. This means that there will be no heat exchange either.

Hence a purely specularly reflecting wall is a very unrealistic model. But often it is a good model to assumes that most of the reflexion is diffuse but a small portion of the molecules are reflected specularly. In the sequel we will however assume that the molecules reflect diffusely with the wall.

3.1   Heat conduction for $Kn >> 1$.**Free molecular flow**

Now we consider two parallel plates with temperatures T_1 and T_2. The space between them is filled with gas of density n. Stationary condition has been established. We consider the case where the Knudsen number is very large. Either because the gas is considerably rarefied or the distance between the plates is very small. In this case we can simply neglect the collisions between the gas molecules. They collide with the walls only.

Wall densitities This means that the distribution function satisfies the collisionless equation, the one particle Liouville equation. The value of $\mathcal{F}$ is carried with the motion of the molecule. Hence the gas molecules are distributed according to two half Maxwellians.

$$\mathcal{F} = \begin{matrix} n_1 \beta_1^3 \pi^{-3/2} \exp(-\beta_1^2 c^2), \ c_{y>0} \\ n_2 \beta_1^3 \pi^{-3/2} \exp(-\beta_2^2 c^2), \ c_{y<0} \end{matrix} \tag{14}$$

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So far, n_1 and n_2 are unknown. Let us first see how they are related to the density n of the gas. We need to calculate for the molecules coming from plate 1 (we introduce $\boldsymbol{\xi} = \beta_1 \mathbf{c}$)

$$
\begin{aligned}
\int_{c_y>0} \mathcal{F}d^3c &= \pi^{-3/2} n_1 \int_{\xi_y>0} \exp(-\xi^2)d^3\xi \\
&= \frac{1}{2}\pi^{-3/2} n_1 \int \exp(-\xi^2)d^3\xi = \frac{n_1}{2}.
\end{aligned}
$$

For the molecules coming from plate 2 we obtain $n_2/2$. Hence

$$
n = \frac{1}{2}(n_1 + n_2).
$$

But there is also the condition that the number of molecules is conserved. This means that the particle current has to vanish, $nv_y = 0$. We can then for a half Maxwellian use (12). As a result we obtain

$$
nv_y = \int c_y \mathcal{F}d^3c = (2\pi)^{-1/2}[n_1\beta_1^{-1} - n_2\beta_2^{-1}].
$$

This has to vanish. This gives us

$$
\frac{n_2}{n_1} = \frac{\beta_2}{\beta_1} = \sqrt{\frac{T_1}{T_2}}.
$$

Or

$$
\begin{aligned}
n_1 &= n\frac{2\sqrt{T_2}}{\sqrt{T_1} + \sqrt{T_2}}, \\
n_2 &= n\frac{2\sqrt{T_1}}{\sqrt{T_1} + \sqrt{T_2}}.
\end{aligned}
\tag{15}
$$

Temperature   What temperature will the gas have under stationary conditions? It is given by

$$
\frac{m}{2}\overline{c'^2} = \frac{3kT}{2}.
$$

As the flow velocity vanishes here, we can skip the prime.

$$
\begin{aligned}
n\overline{c'^2} &= \int c^2 \mathcal{F}d^3c = \pi^{-3/2}[n_1\beta_1^{-2}\int_{\xi_y>0}\xi^2\exp(-\xi^2)d^3\xi \\
&\quad +n_2\beta_2^{-2}\int_{\xi_y<0}\xi^2\exp(-\xi^2)d^3\xi] \\
&= \pi^{-3/2}\frac{1}{2}(n_1\beta_1^{-2} + n_2\beta_2^{-2})\int \xi^2\exp(-\xi^2)d^3\xi.
\end{aligned}
$$

Introducing polar coordinates and then by partial integration we have

$$\int \xi^2 \exp(-\xi^2)d^3\xi$$

$$= 4\pi \int_0^\infty \xi^4 \exp(-\xi^2)d\xi$$

$$= 4\pi \frac{3}{2} \int_0^\infty \xi^2 \exp(-\xi^2)d\xi$$

$$= 4\pi \frac{3}{4} \int_0^\infty \exp(-\xi^2)d\xi$$

$$= 3\pi \frac{1}{2} \int_{-\infty}^\infty \exp(-\xi^2)d\xi = \frac{3\pi^{3/2}}{2}.$$

Hence,

$$n\overline{c'^2} = \frac{3}{4}(n_1\beta_1^{-2} + n_2\beta_2^{-2})$$

$$= \frac{3k}{m} \frac{T_1\sqrt{T_2} + T_2\sqrt{T_1}}{\sqrt{T_1} + \sqrt{T_2}}.$$

$$\frac{m}{2}\overline{c'^2} = \frac{3k}{2} \frac{T_1\sqrt{T_2} + T_2\sqrt{T_1}}{\sqrt{T_1} + \sqrt{T_2}} = \frac{3}{2}kT.$$

This gives us

$$T = \frac{T_1\sqrt{T_2} + T_2\sqrt{T_1}}{\sqrt{T_1} + \sqrt{T_2}} = \sqrt{T_1 T_2}. \tag{16}$$

Heat current   Let us finally calculate the most interesting quantity, the heat current.

$$q_y = n\frac{\overline{mc'^2}}{2}c'_y$$

$$= \frac{m}{2} \int c^2 c_y \mathcal{F} d^3 c$$

$$= \frac{m}{2}\pi^{-3/2}[n_1\beta_1^{-3} \int_{\xi_y>0} \xi^2 \xi_y \exp(-\xi^2)d^3\xi$$

$$+ n_2\beta_2^{-3} \int_{\xi_y<0} \xi^2 \xi_y \exp(-\xi^2)d^3\xi]$$

To calculate the integrals here, we introduce polar coordinates with polar axis along the y-axis

$$\int_{\xi_y>0} \xi^2 \xi_y \exp(-\xi^2)d^3\xi$$

$$= 2\pi \int_0^{\pi/2} d\theta \int_0^\infty \xi^5 d\xi \cos\theta \exp(-\xi^2)\sin\theta.$$

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Here,

$$\int_0^{\pi/2} \cos\theta \sin\theta d\theta = \frac{1}{2}$$

and

$$\int_0^\infty \xi^5 d\xi \exp(-\xi^2) = \frac{4}{2}\frac{2}{2}\int_0^\infty \xi d\xi \exp(-\xi^2) = 1.$$

The term with $\xi_y < 0$ gives the same result with a minus sign. As a result we obtain

$$\begin{aligned}
q_y &= \frac{1}{2}m\pi^{-1/2}[n_1\beta_1^{-3} - n_2\beta_2^{-3}] = \frac{nm}{\sqrt{\pi}}\frac{1}{\beta_1 + \beta_2}(\beta_1^{-2} - \beta_2^{-2}) \\
&= nk\sqrt{\frac{2k}{\pi m}}\frac{\sqrt{T_1 T_2}}{\sqrt{T_1} + \sqrt{T_2}}(T_1 - T_2).
\end{aligned} \tag{17}$$

4 Small Knudsen number

4.1 A simple relaxation model

Now we consider the case where the mean free path ℓ is much smaller than a characteristic length of the configuration. This means that the Knudsen number $Kn = \ell/L << 1$. To have a physical picture of what is going on we assume that all molecules collide at the same time and then move without collisions for a time τ. Then they collide again. At the collisions the molecules are thermalized, so they follow a local Maxwellian.

4.1.1 Temperature gradient

For simplicity we consider a situation with a temperature gradient in the x-direction but no flow. The pressure has to be constant, otherwise the gas would start flowing. We solve n from the gas law to obtain $n = p/kT$. The Maxwellian is (remember that T is a function of x)
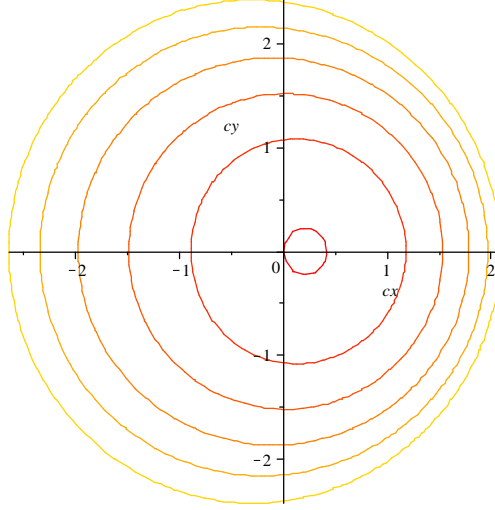
$$\mathcal{F}_0 = n(\frac{m}{2\pi kT})^{3/2}\exp(-\frac{mc^2}{2kT}) = \frac{p}{k}(\frac{m}{2\pi})^{3/2}T^{-5/2}\ \exp(-\frac{mc^2}{2kT}).$$

As long as no new collisions have occurred the molecules carry the value of the distribution function with them. Hence at time $\tau/2$

$$\mathcal{F}(\mathbf{r}, \mathbf{c}) = \mathcal{F}_0(\mathbf{r} - \mathbf{c}\frac{\tau}{2}, \mathbf{c}).$$

$\mathbf{c}\tau/2 \sim \ell$ and T changes little in a distance ℓ. So

$$\begin{aligned}
\mathcal{F}(\mathbf{r}, \mathbf{c}) &\approx \mathcal{F}_0(\mathbf{r}, \mathbf{c}) - \frac{\tau}{2}\mathbf{c}\cdot\frac{\partial\mathcal{F}_0}{\partial\mathbf{r}} \\
&= \mathcal{F}_0(\mathbf{r}, \mathbf{c})(1 - \frac{\tau}{2}c_x\frac{\partial}{\partial x}\ln\mathcal{F}_0).
\end{aligned}$$

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We have

$$\ln \mathcal{F}_0 = -\frac{5}{2}\ln T - \frac{mc^2}{2kT} + const.$$

$$\frac{\partial}{\partial \mathbf{x}}\ln \mathcal{F}_0 = (\frac{mc^2}{2kT^2} - \frac{5}{2T})\frac{\partial T}{\partial x} = (\beta^2 c^2 - \frac{5}{2})\frac{1}{T}\frac{\partial T}{\partial x}.$$

$$\mathcal{F}(\mathbf{r},\mathbf{c}) \approx \mathcal{F}_0(\mathbf{r},\mathbf{c})(1 - \frac{\tau}{2T}\frac{\partial T}{\partial x}c_x(\beta^2 c^2 - \frac{5}{2})). \tag{18}$$

This is the time average of the distribution function. We see that it is no more a local Maxwellian. The figure shows equal density curves for the distribution function in the xy-plane. The variables are the dimensionless velocity components βc_x and βc_y.

Note that

$$\tau c_x \frac{1}{T}\frac{\partial T}{\partial x} \sim \frac{\ell}{L} = Kn$$

and the factor $\beta^2 c^2 - \frac{5}{2} \sim 1$.

It is a good exercise to show that the new term in the distribution function does not contribute to n. Further that $v_x = 0$. To show this it could be convenient to introduce spherical polar coordinates with axis along the x-axis, so that $c_x = c\cos\theta$.

What is interesting, however, is that this distribution function gives a heat current. For symmetry reasons it is clear that the heat current also will be along the x-axis. The zero order terrm does not contribute.

$$q_x = \frac{m}{2}\int c'^2 c'_x \mathcal{F}d^3c = -\kappa\frac{\partial T}{\partial x},$$

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where

$$\kappa = \int \frac{\tau}{2T}(\beta^2 c'^2 - \frac{5}{2})c'^2 c_x'^2 \mathcal{F}_0 d^3 c \tag{19}$$

is the heat conductivity.

$$
\begin{aligned}
q_x &= -\frac{m}{2}n(\frac{\beta}{\sqrt{\pi}})^3 \frac{\tau}{2T}\frac{\partial T}{\partial x}\int \exp(-\beta^2 c^2)(\beta^2 c^2 - \frac{5}{2})c^2 c_x^2 d^3 c \\
&= -\frac{\tau mn}{4T}\frac{\partial T}{\partial x}\beta^{-4}\pi^{-3/2}\int \exp(-\xi^2)(\xi^2 - \frac{5}{2})\xi^2 \xi_x^2 d^3\xi
\end{aligned}
$$

Here we have changed to dimensionless velocities $\xi_i = \beta c_i$. Use spherical polar coordinates with polar axis along the x-axis. Then $\xi_x = \xi\cos\theta$

$$
\begin{aligned}
I &= \pi^{-3/2}\int \exp(-\xi^2)(\xi^2 - \frac{5}{2})\xi^2 \xi_x^2 d^3\xi \\
&= \pi^{-3/2}\int_0^\infty \exp(-\xi^2)(\xi^2 - \frac{5}{2})\xi^6 d\xi \int_0^\pi \cos^2\theta\sin\theta d\theta 2\pi \\
&= \frac{4}{3\sqrt{\pi}}\int_0^\infty \exp(-\xi^2)(\xi^8 - \frac{5}{2}\xi^6)d\xi.
\end{aligned}
$$

But (for $\mathrm{Re}\mu > -1$, so that the integral to the right is convergent)

$$\int_0^\infty \exp(-\xi^2)\xi^\mu d\xi = \frac{\mu - 1}{2}\int_0^\infty \exp(-\xi^2)\xi^{\mu-2}d\xi$$

and

$$
\begin{aligned}
\int_0^\infty \exp(-\xi^2)\xi d\xi &= \frac{1}{2}, \\
\int_0^\infty \exp(-\xi^2)\xi d\xi &= \int_{-\infty}^\infty \exp(-\xi^2)d\xi = \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

We find that

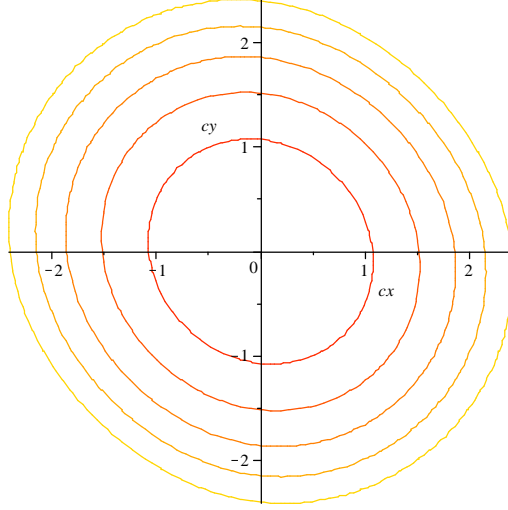$$I = \frac{4}{3\sqrt{\pi}}(\frac{7}{2} - \frac{5}{2})\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi} = \frac{5}{2}.$$

giving

$$q_x = -\frac{5\tau mn}{8T}\frac{\partial T}{\partial x}\beta^{-4} = -\frac{5\tau nkT}{2}\frac{1}{T}\frac{\partial T}{\partial x}(\frac{kT}{m}). \tag{20}$$

Here,

$$\sqrt{\frac{kT}{m}} \sim c_{thermal}$$

$$\tau\sqrt{\frac{kT}{m}}\frac{1}{T}\frac{\partial T}{\partial x} \sim \frac{\ell}{L},$$

$$nkT \sim e.$$

The heat current transports thermal energy with the velocity which is $\sim (\ell/L)c_{thermal}$. The heat conductivity is

$$\frac{5k}{2m}\tau p.$$

4.1.2 Shearing

The picture here is of course quite simplified but can give a feeling for how the distribution function deviates slightly from a local Maxwellian due to the small but non-zero mean free path. One can use a similar reasoning for a shear flow $v_x(y)$. The distribution function is then found to be

$$\mathcal{F} \approx \mathcal{F}_0 [1 - \frac{\tau}{2}\beta^2 c_x' c_y' \frac{dv_x}{dy}]. \tag{21}$$

The distribution function is non-Maxwellian in a characteristic way. Equal density curves with dimensionless velocity components $\beta c_x, \beta c_y$ on the axes.

In this case we obtain a shear stress

$$\sigma_{xy} = -m \int c_{x'} c_y' \mathcal{F} d^3 c = \mu \frac{\partial v_x}{\partial y}.$$

Here

$$\mu = mn \int \frac{\tau}{2} c_{x'}^2 c_y'^2 \mathcal{F}_0 d^3 c \tag{22}$$

is the viscosity

$$\sigma_{xy} = \frac{1}{8}\tau n \beta^{-2} m \frac{\partial v_x}{\partial y} = \frac{1}{4}\tau n k T \frac{\partial v_x}{\partial y} = \frac{1}{4}\tau \frac{\partial v_x}{\partial y} p$$

Here, $\tau dv_x/dy$ is a small dimensionless number $\sim Kn$.

The viscosity is

$$\frac{1}{4}\tau p$$

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It is easy to show that the stress tensor only has a pressure $p = nkT$ besides this shear stress.

For a general velocity field the result is

$$\sigma_{ij} = -p\delta_{ij} + \frac{1}{4}\tau[\frac{1}{2}(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}) - \frac{1}{3}\frac{\partial v_k}{\partial x_k}\delta_{ij}]p. \tag{23}$$

4.2 The Chapman-Enskog expansion

To treat the case with small Knudsen number in a more rigorous way we first introduce dimensionless variables denoted with a tilde

$$x_i = \widetilde{x}_i L, \; t = \widetilde{t}t_c, \; c_i = \widetilde{c}_i C, \; F_i = \widehat{F}_i LC^2 m, n = \widetilde{n}N, \; d\sigma = d^2 d\widetilde{\sigma}.$$

The Boltzmann equation then takes the form (here we have skipped the tildes)

$$\varepsilon(\frac{L}{Ct_c}\partial_t + c_i\partial_{x_i} + \frac{F_i}{m}\partial_{c_i})\mathcal{F} = \int (\mathcal{F}(\mathbf{c}_1^*)\mathcal{F}(\mathbf{c}^*) - \mathcal{F}(\mathbf{c}_1)\mathcal{F}(\mathbf{c}))d\sigma c_{rel}d^3 c_1$$

where

$$\varepsilon = \frac{1}{d^2 NL} = \frac{\ell}{L} = Kn. \tag{24}$$

Here, $L/t_c C$ is a Mach number. If it is of the order of 1 or less, we can simply choose $t_c = L/C$ and $L/t_c C = 1$.

It is convenient to introduce a bilinear operator, such that

$$J(\mathcal{F}, \mathcal{G}) = \int \frac{1}{2}(\mathcal{F}(\mathbf{c}_1^*)\mathcal{G}(\mathbf{c}^*) + \mathcal{F}(\mathbf{c}^*)\mathcal{G}(\mathbf{c}_1^*) - \mathcal{F}(\mathbf{c}_1)\mathcal{G}(\mathbf{c}) - \mathcal{F}(\mathbf{c})\mathcal{G}(\mathbf{c}_1))d\sigma c_{rel}d^3 c_1. \tag{25}$$

The collision integral is then $J(\mathcal{F}, \mathcal{G})$.

It is important that for small Knudsen number, the small parameter multiplies the derivatives. This is typical of a singular perturbation. Now we assume an expansion

$$\mathcal{F} = \mathcal{F}_0 + \varepsilon\mathcal{F}_1 + ... \tag{26}$$

If we insert it into the Boltzmann equation we obtain to zero order

$$J(\mathcal{F}_0, \mathcal{F}_0) = 0. \tag{27}$$

This means that the zero order distribution function is a local Maxwellian.

We extend the collision integral as a symmetric bilinear expression by

$$2J(\mathcal{F}, \mathcal{G}) = 2J(\mathcal{G}, \mathcal{F}) = J(\mathcal{F} + \mathcal{G}, \mathcal{F} + \mathcal{G}) - J(\mathcal{F}, \mathcal{F}) - J(\mathcal{G}, \mathcal{G})$$

Then to next order we find

$$(\partial_t + c_i\partial_{x_i} + \frac{F_i}{m}\partial_{c_i})\mathcal{F}_0 = 2J(\mathcal{F}_0.\mathcal{F}_1). \tag{28}$$

Here we know that the right hand side when multiplied with $1, \mathbf{c}, c^2$ vanishes. Hence the same holds for the left hand side. But this is just the conservations

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laws for the distribution function $\mathcal{F}_0$. These are the Euler equations. From these the time derivatives of $n, \mathbf{v}, T$ can be expressed in terms of spatial derivatives. Thus the left hand side of (28) can be written in terms of the spatial derivatives of $T, \mathbf{v}$.

$$\mathcal{F}_0[(\beta^2 c'^2 - \frac{5}{2})\beta c_i' \frac{T_{,i}}{T} + 2\beta^2 c_i' c_j' v_{<i,j>}]$$

Here

$$\begin{aligned} \mathbf{c}' &= \mathbf{c} - \mathbf{v} \\ v_{<i,j>} &= \frac{1}{2}(v_{i,j} + v_{j,i}) - \frac{1}{3}v_{k,k}\delta_{ij} \end{aligned}$$

$$\mathcal{F}_0 + \varepsilon\mathcal{F}_1 = \mathcal{F}_0\{1 - \frac{1}{n}[(A(\beta^2 c'^2)c_i'\frac{T_{,i}}{T} + 2\beta^2 c_i' c_j' B(\beta^2 c'^2)v_{<i,j>}]\} + O(\varepsilon^2) \quad (29)$$

We see that the result can be obtained from the simple relaxation model by the replacement

$$\frac{\tau}{2} \to \frac{A(\beta^2 c'^2)}{n} \quad (30)$$

for the temperature gradient and

$$\frac{\tau}{2} \to \frac{2B(\beta^2 c'^2)}{n} \quad (31)$$

for the shearing. This means that the relaxation times depend on the dimensionless speed and is different for temperature gradient and shear.

When we make the replacement $\tau/2 \to A/n$ in (19) we find the heat conductivity and the replacement $\tau/2 \to 2B/n$ in (22) gives the viscosity. One important thing that has been achieved by the Chapman-Enskog expansion is a relation between heat conductivity and viscosity in fluid dynamics with the cross section of the molecular interactions.

(Really, the fluid dynamic variables $n, \mathbf{v}, T$ in $\mathcal{F}_0$ are $n_0, \mathbf{v}_0, T_0$ and from (28) $\mathcal{F}_1$ is given in terms of $n_0, \mathbf{v}_0, T_0$ and their first spatial derivatives plus a term that gives zero in $J(\mathcal{F}_0.\mathcal{F}_1)$. From $\mathcal{F}_0 + \varepsilon\mathcal{F}_1$ one calculates the fluid variables, they are $n_{(1)} = n_0 + \varepsilon n_1$ and similarly for $\mathbf{v}, T$. Expressing $\mathcal{F}_0 + \varepsilon\mathcal{F}_1$ in terms of $n_{(1)}, \mathbf{v}_{(1)}, T_{(1)}$ one obtains (29).

If one does not make this summing of the fluid variables, the resulting fluid dynamics equations will not be valid for times characteristic of diffusion as the Navier-Stokes equations are.)

# 5 Knudsen layer and kinetic boundary conditions for Navier-Stokes

When the mean free path is small compared to typical dimensions of the geometry, we have a small Knudsen number and the Boltzmann equation can be

approximated with the Navier-Stokes equations. This is however not fully true. Close to a wall, the distance to the wall is the most important length scale. When it is of the order of the mean free path, the effective Knudsen number is of order 1 and no longer small. As a consequence, there will be a layer of thickness ℓ where the Navier-Stokes equations don't apply. This layer is called a *Knudsen layer* named after the Danish physicist *Martin Knudsen* (1871-1949).

5.1   Velocity slip

We consider a shearing flow close to a wall. According to the usual non-slip boundary condition the velocity is given by $v_x = (\partial v_x/\partial y)y$ and vanishes at the wall given by $y = 0$. We take a point with a distance to the wall which is small compared to ℓ. Now the point is that the molecules with velocities pointing to the wall have travelled a mean free path so they have a mean velocity of the order $(\partial v_x/\partial y)\ell$. We assume the wall reflects diffusely (possibly with a small contribution of specular reflexion). So the outgoing molecules will have an average velocity which is zero (or small). The average for all molecules will be of the order of half $(\partial v_x/\partial y)\ell$. This means that the velocity close to the wall will not be zero. We have a *velocity slip*.

This can be in a rigorous way. Close to the wall one rescales the Boltzmann equation using ℓ as a length scale normal to the wall. The resulting kinetic equation can be solved numerically. The result for hard spheres is this. The distance to the wall is measured in mean free paths. Data are taken from Sone, Kinetic Theory and Flud Dynamics, 2002. The full solution can then be found
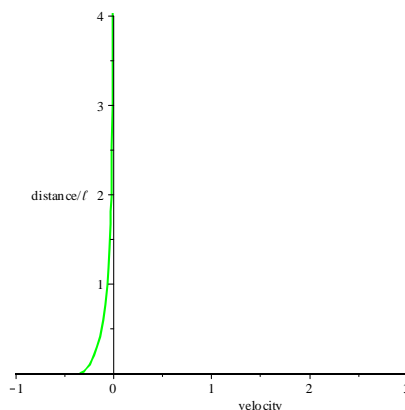
by adding the Navier-Stokes solution. In this way the boundary condition for $\mathbf{v}$ is also found. We see that the tangential velocity of the Navier-Stokes solution vanishes when we extrapolate the curve of the order one mean free path into the wall
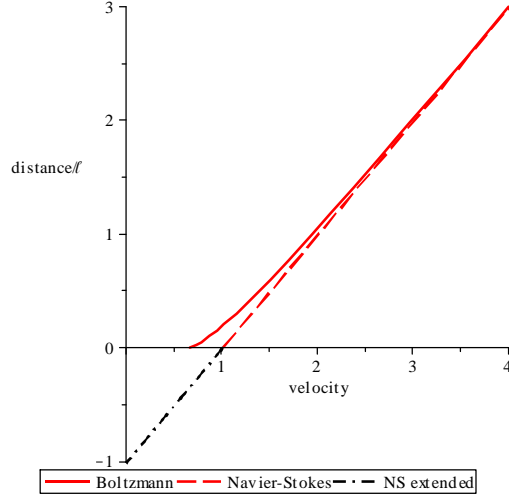
5.2 Temperature jump

The temperature has an entirely analogous behaviour. The Navier-Stokes-Fourier solution will attain the wall temperature only when extrapolated of the order of one mean free path into the wall. And there is a Knudsen layer contribution.

5.3 Thermal creep

There is however also another effect of the Knudsen layer. This is when there is a temperature gradient along the wall. The same temperature gradient will be established in the gas close to the wall. Let us consider a point close to the wall compared to the mean free path. The molecules with velocities pointing to the wall have travelled a mean free path or mean free time from the gas. We already know their distribution, see . But the molecules leaving the wall have all arrived from a point very close to the point we consider. This means that they will follow the local Maxwellian of the wall. So the total distribution function will look like this (the normal into the gas is in the y-direction).It is given by

$$\mathcal{F}(\mathbf{r}, \mathbf{c}) \approx \mathcal{F}_0(\mathbf{r}, \mathbf{c})\{ \begin{matrix} (1 - \frac{\tau}{2T} \frac{\partial T}{\partial x} c_x(\beta^2 c^2 - \frac{5}{2})), c_y < 0 \\ 1, c_y > 0 \end{matrix} . \tag{32}$$

The point is now that this distrubution gives rise to a shear stress. We know it is calculated as the integral of $-c_x c_y \mathcal{F}$. It is clear that the Maxwellian term, will not give rise to shear. So what remains is

$$\sigma_{xy} = \frac{\tau}{2T} \frac{\partial T}{\partial x} \int_{c_y < 0} c_x c_y \mathcal{F}_0(\mathbf{r}, \mathbf{c}) c_x (\beta^2 c^2 - \frac{5}{2})) d^3 c.$$

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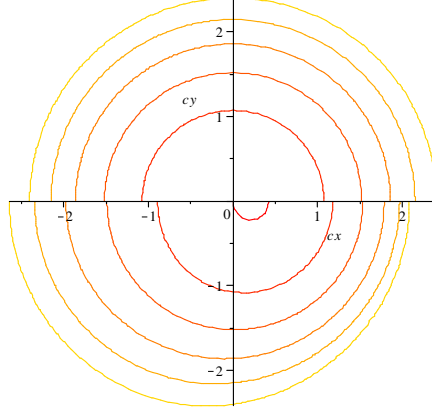
We choose polar coordinates with axis along the y-axis. Then $c_y = c\cos\theta, c_x = c\sin\theta\sin\varphi, d^3c = c^2 dc\sin\theta d\theta d\varphi$. Introducing dimensionless variables, the integral is proportional to

$$\int_0^\infty (\xi^7 - \frac{5}{2}\xi^5)\exp(-\xi^2)d\xi \int_{\pi/2}^\pi \sin^3\theta\cos\theta d\theta \int_0^{2\pi} \sin^2\varphi d\varphi$$
$$= (\frac{6}{2}\cdot\frac{4}{2}\cdot\frac{2}{2} - \frac{5}{2}\cdot\frac{4}{2}\cdot\frac{2}{2})\frac{\sqrt{\pi}}{2}(-\frac{1}{4})\pi = -\frac{\pi^{3/2}}{4}.$$

$$\sigma_{xy} = -\frac{\tau}{8T}\frac{\partial T}{\partial x}\beta^{-3}.$$

At a distance of the order mean free path from the wall, most of the molecules come from the gas, so there we have simply the distribution function for a temperature gradient and no shear stress. So the derivative $d\sigma_{xy}/dy$ will be positive, giving the gas an acceleration in the positive x-direction and set up a flow in that direction. This is the phenomen of *thermal creep*.We estimate this velocity. We assume it to be v in the positive x-direction. This means that the local Maxwellian $\mathcal{F}_0(\mathbf{r}, \mathbf{c})$ should be replaced by $\mathcal{F}_0(\mathbf{r}, \mathbf{c} - \mathbf{v})$. $\mathbf{v}$ is small, so

$$\mathcal{F}_0(\mathbf{r}, \mathbf{c} - \mathbf{v}) \approx \mathcal{F}_0(\mathbf{r}, \mathbf{c})[1 + \frac{mv(c_x - v_x)}{kT}]$$
$$\approx \mathcal{F}_0(\mathbf{r}, \mathbf{c})[1 + 2\beta^2 c_x v_x].$$

So the new distribution function is given by the replacement

$$-\frac{\tau}{2T}\frac{\partial T}{\partial x}(\beta^2 c^2 - \frac{5}{2}) \to -\frac{\tau}{2T}\frac{\partial T}{\partial x}(\beta^2 c^2 - \frac{5}{2}) + 2\beta^2 c_x v_x.$$

For this to give the shear stress zero we find

$$-\frac{\tau}{2T}\frac{\partial T}{\partial x}\frac{\sqrt{\pi}}{2} + \beta^2 v_x\sqrt{\pi} = 0.$$

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This gives us

$$v_x = \frac{1}{\beta^2} \frac{\tau}{4T} \frac{\partial T}{\partial x} = \frac{k\tau}{2m} \frac{\partial T}{\partial x}$$

We have

$$v_x \sim \frac{\ell}{L} \sqrt{\overline{c'^2}}.$$

5.3.1   Thermophoresis

An interesting application is that of a body in a gas with a temperature gradient. There will then be a temperature gradient also in the body unless it has an extremely high thermal conductivity. Due so thermal creep the gas around the body will start streaming in the direction of increasing temperature. As a consequence the body will move in the opposite direction. It will move from hotter to colder gas. This is the phenomen of *thermophoresis*. This phenomenon can be seen when a candle is burning close to a cold wall. Soot particles will move from the candle to the wall and get stuck there.

Here the temperature gradient is to the left (green arrow). A flow will start along the boundary of the body. It is given by the blue arrows. As a result the body will move in the opposite direction (black arrow).
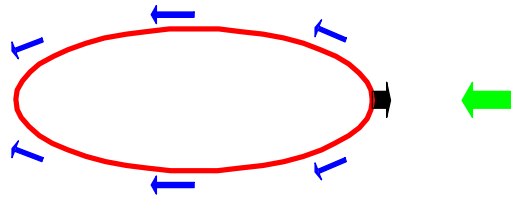
5.3.2   Radiometer

Another example is the *radiometer*. It is used to measure electromagnetic radiation, in particular infrared light.

The blades are black on one side and metallic on the other side. The glass container is filled with rarefied air. If the container had a perfect vacuum one
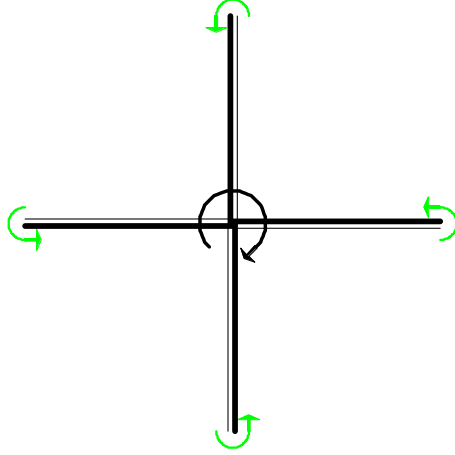
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Figure 1: The radiometer seen from above. The gas flow is in the direction from the colder to the hotter sides and the radiometer is rotating in the opposite direction

would expect the photons to be absorbed on the black sides and reflected on the metallic sides. As a result a rotation would set up with the black side coming first. But in a radiometer, there is some air and the mechanism is quite different. The black sides will have a higher temperature than the metallic sides. Hence, at the tip of a blade there will be a temperature gradient pointing from the metallic side to the black side. A thermal creep will set up, the gas will flow from the metallic side to the black side. This will give a back reaction on the blades and the radiometer will start to move in the opposite direction.

5.4   Boundary conditions for Navier-Stokes-Fourier

The boundary conditions for the Navier-Stokes-Fourier solution are ($\mathbf{n}$ is the normal into the gas, T_w is the wall temperature)

$$v_i - n_i n_j v_j - \xi_v[v_{i,j} + v_{j,i} - 2n_i v_{j,k} n_k]n_j \quad = \quad \mu_T(T_{,i} - n_i n_j T_{,j}), \quad (33)$$
$$T - \xi_T n_i T_{,i} \quad = \quad T_w$$

Here, ξ_v, ξ_T are coefficients $\sim \ell$ and $\mu_T \sim \frac{\ell}{T}\sqrt{c'^2} \sim \frac{\ell c_s}{T}$. ξ_v gives the velocity jump and ξ_T the temperature jump. μ_T is the coefficient of thermal creep.

If $v_z(x,y)$ is the only non-zero component of velocity, as in a pipe flow, the boundary conditions are simply

$$v_z - \xi_v \frac{\partial v_z}{\partial n} \quad = \quad \mu_T \frac{\partial T}{\partial z}, \quad (34)$$
$$T - \xi_T \frac{\partial T}{\partial n} \quad = \quad T_w$$

6 Flows driven by a temperature difference

A tube with circular cross section and with a temperature gradient. Solving the Navier-Stokes equations we obtain

$$v = \frac{1}{4\eta}\frac{dp}{dz}r^2 + A.$$

The boundary condition is (the outward normal of the surface is in the negative radial direction)

$$v + \xi_v \frac{dv}{dr} = \mu_T \frac{dT}{dz}$$

or

$$\frac{1}{4\eta}\frac{dp}{dz}R^2 + A + \xi_v \frac{1}{4\eta}\frac{dp}{dz}2R = \mu_T \frac{dT}{dz}.$$

Hence,

$$
\begin{aligned}
A &= -\frac{1}{4\eta}\frac{dp}{dz}(R^2 + 2R\xi_v) + \mu_T \frac{dT}{dz} \\
&\approx -\frac{1}{4\eta}\frac{dp}{dz}(R + \xi_v)^2 + \mu_T \frac{dT}{dz}.
\end{aligned}
$$

as $\xi_v << R$.

$$v = -\frac{1}{4\eta}\frac{dp}{dz}[(R + \xi_v)^2 - r^2] + \mu_T \frac{dT}{dz}. \tag{35}$$

The first correction involving ξ_v means that the flow is the same as if the radius of the tube were $R + \xi_v$, of the order of a mean free path larger then the real radius R. The second correction is the one we will consider now. The total mass flux is

$$Q = -\frac{\pi\rho}{8\eta}\frac{dp}{dz}[2(R + \xi_v)^2 R^2 - R^4] + \pi\rho\mu_T \frac{dT}{dz}R^2$$

6.1 Closed tube

We now assume that the tube is closed at both ends. We will show that a pressure gradient will set up. As the tube is closed $Q = 0$. We can then neglect ξ_v and obtain

$$\frac{dp}{dz} = \frac{8\eta\mu_T}{R^2}\frac{dT}{dz} \tag{36}$$

The pressure will increase in the direction of increasing temperature to counterbalance the thermal creep flow caused by the temperature gradient. As an estimate (recall that $\eta \sim \rho c_s \ell$ and $\mu_T \sim \ell c_s/T$; c_s is the speed of sound)

$$\triangle p \sim \rho c_s^2 (\frac{\ell}{R})^2 \frac{\triangle T}{T}.$$

But $mc_s^2 \sim k_B T/m \sim \rho c_s^2$.

$$\frac{\triangle p}{p} \sim (\frac{\ell}{R})^2 \frac{\triangle T}{T}.$$

So the pressure difference is of second order in the Knudsen number based on the radius. A thinner tube will create a larger pressure difference.

In this case, the velocity profile is

$$v = \mu_T \frac{dT}{dz}(\frac{2r^2}{R^2} - 1) \tag{37}$$

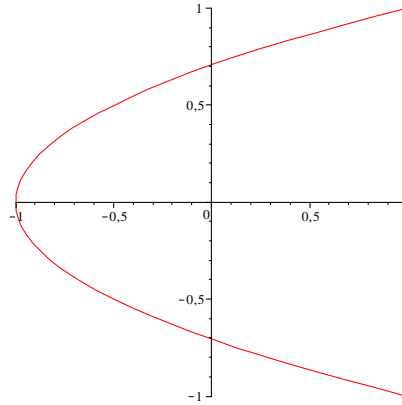Close to the walls, the flow is in the direction of the temperature gradient, but

Figure 2: Velocity profile at equilibrium due to thermal gradient

in the middle of the tube it is in the opposite direction.

6.2   Two connected tubes with different radii

Now we assume that two tubes of different radii R_1 and R_2 are connected in their end points to form a closed system as shown in the picture. But let us first solve for dp/dz, which we want to eliminate

$$\frac{dp}{dz} = 8\eta\mu_T \frac{dT}{dz} \frac{R^2}{[2(R+\xi_v)^2 R^2 - R^4]} - \frac{8\eta}{\pi\rho} \frac{1}{2(R+\xi_v)^2 R^2 - R^4} Q$$

For the first tube, R is replaced by R_1. For the second tube R is replaced by R_1 and Q by $-Q$. We take the difference of the two equations to find the relation between Q and dT/dz. But R_1, R_2 and $|R_2 - R_1|$ are much larger than the mean free path, so we can neglect ξ_v. As a result we find

$$Q = \rho\mu_T \frac{dT}{dz}\pi R_1^2 q(\frac{R_1}{R_2}), \tag{38}$$

$$q(x) = \frac{1 - x^2}{1 + x^4}.$$

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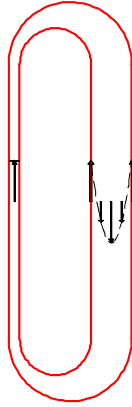
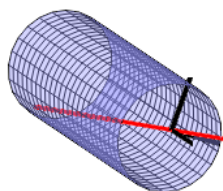The function q is plotted here. As expected, when the two radii are equal, so
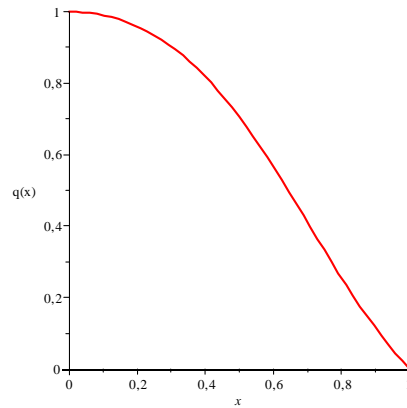
that $x = 1$, the flux is zero. When $R_2 >> R_1$ the flux is

$$Q \approx \rho \mu_T \frac{dT}{dz} \pi R_1^2.$$

7 Free molecular flow

We have earlier considered the heat transfer between two plates for a very high Knudsen number. Now we consider another important example.

7.1 **Flow in a tube at** $Kn >> 1$

We consider a circular tube with radius R. We assume a pressure gradient but no temperature gradient. And we consider the case where $\ell >> R$. This means that the only collisions are those with the wall of the tube. The red line is the motion of a molecule. It starts at the wall.

We then know that the value of the distribution function $\mathcal{F}$ is constant when we follow the motion of a molecule, until it collides with the wall. Let us consider a point with the distance r from the axis. We choose cartesian coordinates such that the point has coordinates $r, 0, z$. We now introduce spherical polar coordinates. The direction of the molecular velocity given by a red line in the figure is given by the angles θ, ϕ. θ is the angle with the z-direction. The dashed line is the projection of the molecular velocity on the xy-plane. The angle ϕ is the angle of the dashed line from the x-axis. To find the value of the distribution function we simply need to find out where the molecule has last collided with the wall. We call the coordinates of that point x_0, y_0, z_0.

The speed of the molecule is c. So the components of the velocity are given by

$$
\begin{aligned}
c_x &= c\sin\theta\cos\phi, \\
c_y &= c\sin\theta\sin\phi, \\
c_z &= c\cos\theta
\end{aligned}
$$

We find, if we assume that the molecule has traveled the time $\triangle t$ since it collided
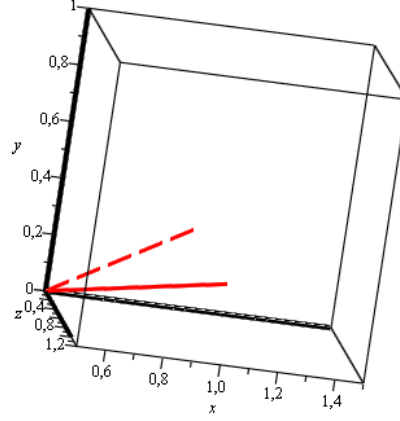
with the wall,

$$
\begin{aligned}
x_0 &= r - c\triangle t \sin\theta \cos\phi, \\
y_0 &= -c\triangle t \sin\theta \sin\phi, \\
z_0 &= z - c\triangle t \cos\theta.
\end{aligned}
$$

Here we eliminate $c\triangle t$ and obtain

$$
\begin{aligned}
x_0 &= r + (z - z_0)\tan\theta\cos\phi, \\
y_0 &= (z - z_0)\tan\theta\sin\phi.
\end{aligned}
$$

But the distance to the axis is then the radius R of the tube

$$
R^2 = r^2 + (z - z_0)^2 \tan^2\theta + 2r(z - z_0)\tan\theta\cos\phi.
$$

The interesting thing is to find out the value of z_0. We write the equation

$$
(R^2 - r^2)\cot^2\theta = (z - z_0)^2 + 2r(z - z_0)\cot\theta\cos\phi
$$

and find (there are two solutions; we pick the one that gives $z_0 < z$ when $0 < \theta < \pi/2$)

$$
\begin{aligned}
z - z_0 &= -r\cot\theta\cos\phi + \sqrt{(R^2 - r^2)\cot^2\theta + r^2\cot^2\theta\cos^2\phi} \\
&= -r\cot\theta\cos\phi + \cot\theta\sqrt{R^2 - r^2\sin^2\phi} \\
&= \cot\theta(-r\cos\phi + \sqrt{R^2 - r^2\sin^2\phi})
\end{aligned}
$$

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The Maxwellian is given by

$$\mathcal{F}_0 = n_w \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \exp(-\beta^2 c^2).$$

So the distrubution function is

$$\mathcal{F}(r, z, c, \theta, \phi) = n_w(z_0) \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \exp(-\beta^2 c^2).$$

Here

$$\beta^2 = \frac{m}{2kT_w}.$$

We have pressure gradient but a constant temperature. From

$$p = knT$$

we have the density gradient

$$\frac{dn}{dz} = \frac{1}{kT} \frac{dp}{dz}.$$

To a sufficient approximation we write

$$n_w(z_0) \approx n_w(z) + \frac{dn_w}{dz}(z_0 - z)$$

So we obtain the distribution function

$$\mathcal{F}(r, z, c, \theta, \phi) \approx \mathcal{F}_0(R, z, c) - \frac{dn_w}{dz} \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \cot\theta(-r\cos\phi + \sqrt{R^2 - r^2\sin^2\phi}) \exp(-\beta^2 c^2).$$

$$(39)$$

Let us now find the velocity macroscopic velocity v_z. To that end we use

$$nv_z = \int \mathcal{F} c_z d^3 c$$

The Maxwellian gives no contribution to the integral. The remaning integral is

$$-\frac{dn_w}{dz} \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \int \exp(-\beta^2 c^2) \cot\theta(-r\cos\phi + \sqrt{R^2 - r^2\sin^2\phi}) c\cos\theta c^2 dc \sin\theta d\theta d\phi$$

The term $-r\cos\phi$ does not contribute. The result is

$$-\frac{dn_w}{dz} \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \int \exp(-\beta^2 c^2) c^3 dc \cot\theta \sqrt{R^2 - r^2\sin^2\phi} \cos\theta \sin\theta d\theta d\phi$$

$$= -\frac{dn_w}{dz} \left(\frac{\beta}{\sqrt{\pi}}\right)^3 \int_0^\infty \exp(-\beta^2 c^2) c^3 dc \int_0^\pi \cos^2\theta d\theta \int_0^{2\pi} \sqrt{R^2 - r^2\sin^2\phi} d\phi.$$

Here we introduce $\eta = \beta^2 c^2$ and obtain

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$$\int \mathcal{F}c_z d^3 c \;=\; -\frac{dn_w}{dz}\left(\frac{\beta}{\sqrt{\pi}}\right)^3 \beta^{-4} \int_0^\infty \frac{1}{2}\exp(-\eta)\eta \, d\eta \frac{\pi}{2} R \int_0^{2\pi}\sqrt{1-\left(\frac{r}{R}\right)^2 \sin^2 \phi}\,d\phi$$

$$=\; -\frac{dn_w}{dz}\frac{1}{\beta\sqrt{\pi}}\frac{1}{4}R\int_0^{2\pi}\sqrt{1-\left(\frac{r}{R}\right)^2 \sin^2\phi}\,d\phi$$

$$=\; -\frac{dn_w}{dz}\frac{1}{\beta\sqrt{\pi}}RE\left(\frac{r}{R}\right)$$

We have introduced the elliptic integral of the second kind

$$E(w)=\int_0^{\pi/2}\sqrt{1-w^2\sin^2\phi}\,d\phi.$$

We also need to calculate n.

$$n=\int \mathcal{F}d^3 c$$

The Maxwellian gives simply $n(z)$, the value of n at the boundary. The remaining term in the integral is

$$-\frac{dn}{dz}\left(\frac{\beta}{\pi}\right)^{3/2}\int \exp(-\beta^2 c^2)c^2\,dc\cot\theta\left(-r\cos\phi+\sqrt{R^2-r^2\sin^2\phi}\right)\sin\theta d\theta d\phi$$

$$=\; -\frac{dn}{dz}\left(\frac{\beta}{\pi}\right)^{3/2}\int \exp(-\beta^2 c^2)c^2\,dc\left(-r\cos\phi+\sqrt{R^2-r^2\sin^2\phi}\right)\cos\theta d\theta d\phi$$

The integral over θ vanishes, so the number density is simply constant over a cross section, $n=n_w$.

We conclude that

$$v_z=-\frac{1}{n}\frac{dn}{dz}\frac{R}{\beta\sqrt{\pi}}E\left(\frac{r}{R}\right). \tag{40}$$

To find the total flux, we need the integral

$$\int_0^1 E(w)w\,dw=\frac{2}{3}.$$

As a result we obtain

$$Q_{fm}=2\pi mn\int v_z r\,dr=-\frac{4}{3}R^3\sqrt{\frac{2\pi kT}{m}}m\frac{dn}{dz}. \tag{41}$$
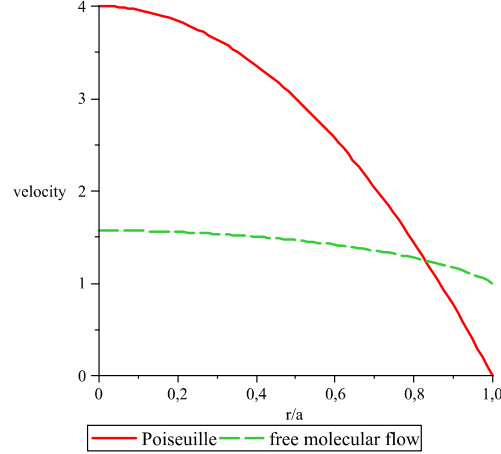
It is interesting here to compare with the Navier-Stokes Poiseuille result

$$Q_{NS}=-mn\frac{\pi R^4}{8\eta}\frac{dp}{dz}.$$

But we know that

$$\eta \;=\; mn\nu \sim mnc_s\ell,$$
$$mc_s^2 \;\sim\; kT$$

Flow in a tube. The dimensionless velocity is defined to give total flux 2 Pi

and use the gas law to obtain

$$Q_{NS} \sim -mn\frac{R^4 kT}{\ell c_s}\frac{dn}{dz} \sim m\frac{R^4}{\ell}m\frac{dn}{dz}c_s$$

In the same way we obtain

$$Q_{fm} \sim -R^3 m\frac{dn}{dz}c_s.$$

We conclude that if we put $\ell \sim R$ in the Navier-Stokes expression we obtain the free molecular flow result. This is reasonable: the diameter of the tube gives an estimate of the distance between two collisions and thus serves as an effective mean free path in this case.

If we consider the integrals for c^2 to obtain the total energy density, we obtain a similar result, it will be a constant. But

$$\begin{aligned} \rho e &= \rho\frac{v^2}{2} + \rho e_0 = \rho\frac{v^2}{2} + \rho\frac{3}{2m}kT, \\ T &= \frac{2m}{3k}(e - \frac{v^2}{2}) \end{aligned}$$

As ρ and ρe are constants over a cross section (independent of r) but $v^2/2$ decreases with the distance from the axis, this means that T increases with the distance from the axis.