Figure 1:

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Continuum Mechanics
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Variation of energy

In an isotropic elastic body, the elastic energy/volume $e = \rho_0 \epsilon$ is a function of E_{ij}. Show that it can be written

$$e(u, u_{i,j}) = \rho_0 \epsilon = \frac{\mu}{2}(u_{i,j}u_{i,j} + u_{i,j}u_{j,i}) + \frac{\lambda}{2}u_{i,j}\delta_{ij}u_{k,l}\delta_{kl}$$

Then consider the total elastic energy

$$E = \int e dv$$

Now make a small change of u_i. Show for an arbitrary function $e(u_i,\, u_{i,j})$ that

$$\begin{aligned} \delta \int e dv &= \int [\frac{\partial e}{\partial u_i}\delta u_i + \frac{\partial e}{\partial u_{i,j}}\delta u_{i,j}] dv \\ &= \int [\frac{\partial e}{\partial u_i} - \frac{\partial}{\partial x_j}(\frac{\partial e}{\partial u_{i,j}})] \delta u_i dv \\ &\quad + \int \frac{\partial e}{\partial u_{i,j}} \delta u_i ds_j \end{aligned}$$

Assume that u_i is given at the boundary. What is then δu_i at the boundary? Show that the surface integral vanishes. Now conclude that the variation of the integral vanishes if and only if

$$\frac{\partial e}{\partial u_i} - \frac{\partial}{\partial x_j}(\frac{\partial e}{\partial u_{i,j}}) = 0$$

as δu_i are arbitrary functions.

Now calculate $\partial e/\partial u_{i,j}$ for the isotropic material. What is it? Find the resulting equations.

Suppose now that the body is heavy so that gravity is important. Then the term

$$-\rho_0 g_i \cdot u_i$$

has to be added to e. Show that the correct equations of equilibrium are obtained also in this case.