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Mathematical methods of mechanics

Home assignment 1 to be handed in Wednesday November 5, 2008

Far-field radiation from localized wall vibrations

We assume that the plane wall $z = 0$ is vibrating, so that its normal velocity is a given function. The vibrations have the angular frequency ω, so all functions have a factor $\exp(-i\omega t)$ and the real part is understood.

a) *The connection between velocity and pressure.*
First of all, show from the linearized Euler equations

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p$$

how the velocity $\mathbf{v}$ can be found from the pressure deviation

$$\rho_0 i\omega \mathbf{v}_\omega = \boldsymbol{\nabla} p_\omega .$$

b) *The Green's function for half-space* $z \geq 0$
We already know the Green function for the whole of space

$$\frac{1}{4\pi} \frac{\exp(ik|\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|}.$$

Now we shall show how the pressure distribution outside the vibrating plane can be calculated from the known vibration, using the Green's function for the half-space. What we need turns out to be a Green's function such that its normal derivative vanishes on the plane. We obtain that Green's function by first taking the usual Green's function for the whole of space

$$\frac{1}{4\pi} \frac{\exp(ik|\mathbf{r} - \mathbf{r}_0|)}{|\mathbf{r} - \mathbf{r}_0|}.$$

It represents the pressure field from a point source of sound at ($z_0 \geq 0$)

$$\mathbf{r_0} = (x_0, y_0, z_0).$$

But the Green's function for the whole space does not satisfy the boundary condition we need on the plane. So we choose a second point source at the mirror point

$$\mathbf{r}_1 = (x_0, y_0, -z_0).$$

We now try two diffent Green's functions

$$g_{\omega\pm}(\mathbf{r},\mathbf{r}_0) = \frac{1}{4\pi}[\frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|} \pm \frac{\exp(ik|\mathbf{r}-\mathbf{r}_1|)}{|\mathbf{r}-\mathbf{r}_1|}].$$

Note that only the original source lies in the half-space. So $g_{\omega\pm}$ satisfy for $\mathbf{r}$ and $\mathbf{r}_0$ in the half-space $z \geq 0$

$$(\boldsymbol{\nabla}^2 + k^2)g_{\omega\pm}(\mathbf{r},\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0).$$

Check now that they satisfy the boundary conditions

$$\begin{aligned}\frac{\partial g_{\omega+}(\mathbf{r},\mathbf{r}_0)}{\partial z} &= 0,\\ g_{\omega-}(\mathbf{r},\mathbf{r}_0) &= 0,\end{aligned}$$

when $\mathbf{r}$ is in the plane $z = 0$.

c) *Finding the pressure distribution in the half-space using Green's function*

The pressure distribution satisfies the Helmholtz equation

$$(\boldsymbol{\nabla}^2 + k^2)p_\omega(\mathbf{r}) = 0.$$

Now use Green's theorem to show that

$$p_\omega(\mathbf{r}) = \int [g_\omega(\mathbf{r},\mathbf{r}_0)\boldsymbol{\nabla}_0 p_\omega(\mathbf{r}_0) - p_\omega(\mathbf{r}_0)\boldsymbol{\nabla}_0 g_\omega(\mathbf{r},\mathbf{r}_0)] \cdot d\mathbf{s}_0.$$

The integration is taken over a large half-sphere with center at the origin and the corresponding equatorial plane of the circle, which is on the plane $z = 0$.

Then use the Sommerfeld radiation condition to show that the integral over the half-sphere can be neglected if the radius of the sphere goes to infinity.

Now, if the pressure distribution on the plane is given, which of the Green's functions is appropriate? If instead the normal derivative is given, which of the Green's functions is the correct one? Show in each case that the boundary condition satisfied by the Green's function on the plane makes one the terms in the integral vanish. In our case, where the wall velocity is given, show that we obtain

$$\begin{aligned}p_\omega(\mathbf{r}) = &- \frac{1}{2\pi}\int\int \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}\frac{\partial p_{gw}}{\partial z}(x_0,y_0,0)dx_0dy_0\\ = &- \frac{i\omega\rho_0}{2\pi}\int\int \frac{\exp(ik|\mathbf{r}-\mathbf{r}_0|)}{|\mathbf{r}-\mathbf{r}_0|}v_\omega(x_0,y_0,0)dx_0dy_0.\end{aligned}$$

Here, $\mathbf{r}_0 = (x_0, y_0, 0)$.

d) *The pressure field from a rigid vibrating disc*

Now assume that a circular disc of radius a in the wall is vibrating, so that v_ω is a constant v for $r_0 = \sqrt{x_0^2 + y_0^2} \leq a$ and zero for $r_0 > a$. At a distance $r >> a$ simplify the result to, using polar coordinates on the plane $z_0 = 0$

$$x_0 = r_0 \cos\phi_0, y_0 = r_0 \sin\phi_0$$

and denoting the angle between $\mathbf{r}$ and the $z-$axis θ

$$p_\omega(\mathbf{r}) = -\frac{i\omega\rho_0}{2\pi} v \frac{\exp(ikr)}{r} \int_0^a \int_0^{2\pi} \exp[-ikr_0 \sin\theta \cos(\phi - \phi_0)] r_0 dr_0 d\phi_0.$$

Check that the integral does not depend on ϕ by changing variables to $\phi - \phi_0$ in the integration to find finally

$$p_\omega(\mathbf{r}) = -\frac{i\omega\rho_0}{k^2} v \frac{\exp(ikr)}{r} F(\theta),$$

where

$$\begin{aligned} F(\theta) &= \frac{k^2}{2\pi} \int_0^a \int_0^{2\pi} \exp[-ikr_0 \sin\theta \cos(\phi_0)] r_0 dr_0 d\phi_0 \\ &= \frac{1}{2\pi \sin^2\theta} \int_0^{ka \sin\theta} \{ \int_0^{2\pi} \exp[-iu\cos(\phi_0)] d\phi_0 \} u du \end{aligned}$$

Try to interpret the result! (The result can be shown to be, but this is not necessary to show)

$$p_\omega(\mathbf{r}) = -i\rho_0 v c k a^2 \frac{J_1(ka\sin\theta)}{ka\sin\theta} \frac{\exp(ikr)}{r}.$$

Hint: write ($\mathbf{n} = \mathbf{r}/r$ is a unit vector in the direction of $\mathbf{r}$)

$$|\mathbf{r} - \mathbf{r}_0| = r\sqrt{1 - \frac{2}{r}\mathbf{n}\cdot\mathbf{r}_0 + (\frac{r_0}{r})^2}.$$

and expand in a Taylor series, as $r_0/r \leq a/r << 1$.