Some exercises for working on the connection of the Nabla notation with the index notation

Lars Söderholm
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1. A scalar field

$$\phi = A\exp(i\mathbf{k}\cdot\mathbf{r})$$

is given. Calculate the gradient $\boldsymbol{\nabla}\phi$ of this vector field. First of all write down $\mathbf{k}\cdot\mathbf{r}$ explicitly as a sum of three terms.

$$\mathbf{r} = (x_1, x_2, x_3)$$

is the radius vector.

Then write it down using the index expression and the Einstein summation convention

$$\mathbf{k}\cdot\mathbf{r}$$

Now write down the gradient of ϕ

a) component by component

$$\boldsymbol{\nabla}\phi = (..,..,..),$$

b) in vector form. What vector should there be on the right hand side?

c) the same expression in component notation

2. Now consider the expression

$$\boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\phi.$$

Is it a scalar, vector or tensor? Write it down explicitly as a sum of three terms. Then write this sum by using the Einstein summation convention.

3. Calculate the expression in the foregoing exercise, using what you found in the first exercise. Write the result a) using vectors b) as an explicit sum of three terms c) using the Einstein summation convention.

4. Now consider the differential operator

$$\triangle = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}$$

You already know that ∇ is the vector

$$\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$$

So if you like you can say that the i-component of ∇ is $\partial/\partial x_i$. Then show that

$$\triangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_i \partial x_i}$$

5. Now calculate $\triangle\phi$ Is the result a scalar, a vector or a tensor? Then write down the result a) using vector notation b) using an explicit sum with three terms c) using the Einstein summation convention

6-9. Try the same problems 1-3,5 for the field

$$\phi = a(\mathbf{r}) \exp(i\mathbf{k} \cdot \mathbf{r}).$$

10. Now we introduce a vector field . Let us write it in component notation

$$b_i = x_k x_k x_i.$$

On the right hand side here. What indices are free and what indices are dummy (or summation)?

11. Could you write down explicitly $x_k x_k$ as a sum of three terms? Could you then express by using the radius vector $\mathbf{r}$?

12. Now it is time to consider the divergence of the vector field

$$\nabla \cdot \mathbf{b}.$$

Is it a scalar, vector or a tensor? Write out this expression as an explicit sum of three terms. Then write it using the Einstein summation convention.

13. For the vector $\mathbf{b}$ calculate

$$b_{i,j}.$$

To start with. Is this a scalar, a vector or a tensor? What indices are free? What indices are dummies? The answer is that it is a tensor, so it has 9 components. Write it down in matrix form. And also calculate all the components from the expression for the vector $\mathbf{b}$. Now try to express it using the Einstein summation convention.

By the way, what is

$$\frac{\partial x_i}{\partial x_j}?$$

14. Now that you have calculated $b_{i,j}$ how do you obtain the divergence? Calculate it. Do that and express the result using vector notation, an explicit sum with three terms and using the Einstein summation convention.

15. As you have now been able to calculate the divergence $\boldsymbol{\nabla} \cdot \mathbf{b}$ you remember that it is a scalar. Now you should be able to calculate the gradient of this scalar to obtain

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{b}).$$

But before doing this. The formula here. Is it a scalar, vector or a tensor?

16. You could write down the divergence $\boldsymbol{\nabla} \cdot \mathbf{b}$ as a sum of three terms or using the Einstein summation convention.

17. The expression in 15 is a vector. Write it in the form

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{b}) = (.., .., ..).$$

Then write down the x_1-component of this equation. And the same for the x_2- and x_3-components. The instead write down the i-component and then use the Einstein summation convention for this expression.

18. Now it is time to really calculate this for the given vector field $\mathbf{b}$. And write the result using vector notation, and explic sum of three components and finally the Einstein summation convention.

19. Let us now instead consider

$$\triangle \mathbf{b} = (\boldsymbol{\nabla} \cdot \boldsymbol{\nabla})\mathbf{b}.$$

Do all the things you did for $\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{b})$ also for this expression.

Some answers

1) $\boldsymbol{\nabla}\phi = iA\exp(i\mathbf{k} \cdot \mathbf{x})(k_1, k_2, k_3) = iA\exp(i\mathbf{k} \cdot \mathbf{x})\mathbf{k}$; $(\boldsymbol{\nabla}\phi)_j = iA\exp(i\mathbf{k} \cdot \mathbf{x})k_j$
2) $\boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\phi = \phi_{,1}\phi_{,1} + \phi_{,2}\phi_{,2} + \phi_{,3}\phi_{,3} = \phi_{,l}\phi_{,l}$

3) -$A^2\exp(i2\mathbf{k} \cdot \mathbf{x})\mathbf{k} \cdot \mathbf{k} = -A^2\exp(i2\mathbf{k} \cdot \mathbf{x})\mathbf{k}^2$
$= - A^2\exp(i2\mathbf{k} \cdot \mathbf{x})(k_1^2 + k_2^2 + k_3^2) = -A^2\exp(i2\mathbf{k} \cdot \mathbf{x})k_lk_l$

5) $\triangle\phi = -(k_1^2+k_2^2+k_3^2)A\exp(i\mathbf{k}\cdot\mathbf{x}) = -\mathbf{k}^2 A\exp(i\mathbf{k}\cdot\mathbf{x}) = -k_l k_l A\exp(i\mathbf{k}\cdot\mathbf{x})$

6-9) $\triangle\phi = a_{,ll}\exp(i\mathbf{k}\cdot\mathbf{x})+2ia_{,l}k_l\exp(i\mathbf{k}\cdot\mathbf{x})-k_l k_l A\exp(i\mathbf{k}\cdot\mathbf{r})$.
16) $\boldsymbol{\nabla}\cdot\mathbf{b} = b_{i,i} = 5x_l x_l = 5\mathbf{r}\cdot\mathbf{r}$

18) $\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{b}) = 10\mathbf{r}; \frac{\partial}{\partial x_i}(b_{l,l}) = b_{l,li} = 10x_i$

19) $\triangle\mathbf{b} = (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\mathbf{b} = 10\mathbf{r}; \frac{\partial^2}{\partial x_l^2}b_i = b_{i,ll} = 10x_i$