Partial Differential Equations with Applications to Wave Theory

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Abstract.

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Preface

This material is used in the course Mathematical Methods of Mechanics for last year undergraduate students at KTH in Stockholm. The aim of the course is to develop mathematical methods and at the same time solve interesting problems in mechanics.

CHAPTER 1

Linear Hyperbolic Waves

1. The wave equation

The strict definition of hyperbolic waves is postponed. It should just be mentioned that the prototype of hyperbolic waves is often taken to be the wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c_0^2 \nabla^2 \phi. \tag{1.1}$$

This equation has a wide range of use in acoustics, elastomechanics and electrodynamics. The field quantity $\phi(x, y, z, t)$ depends on the problem studied. As we shall see the quantity c_0 is the propagation velocity of the wave described by (1.1). A physical derivation of (1.1) will now be given.

1.1. Derivation of the wave equation in fluid acoustics. We often observe fluids, which are more or less at rest. The very fact that we observe this state of equilibrium, implies that it is a stable one. If we consider a particle for the moment, we know that a stable equilibrium means that the particle will oscillate around this point, when slightly disturbed. The fluid, when slightly disturbed, will also tend back towards the state of rest. But also, pressure connects neighbouring fluid elements, so there will be some sort of coupled oscillations of the fluid elements. This is the origin of the mechanism of sound waves.

For these, except if they are of very low frequency, heat conduction can be neglected. And except if they are of very high frequency, viscosity can also be neglected. The equation of continuity of the fluid is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = 0 \tag{1.2}$$

and the Euler equation is

$$\rho[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v}] = -\boldsymbol{\nabla} p. \tag{1.3}$$

The fluid density is ρ, the fluid velocity is $\mathbf{v}$ and the fluid pressure is p.
Sound waves are usually at constant entropy. It is well-known that when heat conduction and viscosity can be neglected, the entropy is constant. So the constitutive equation is

$$p = p(\rho, s_0), \tag{1.4}$$

where s_0 is the value of the constant entropy. In the sequel we shall just write $p(\rho)$.

In particular, for an ideal gas, with constant specific heats,

$$\frac{p}{p_0} = (\frac{\rho}{\rho_0})^\gamma, \tag{1.5}$$

where $\gamma = c_p/c_v$ and where p_0, ρ_0 are the fluid pressure and density at equilibrium. For two-atomic gases $\gamma = 7/5 = 1.4$ except for very low and very high temperatures. For monatomic gases, $\gamma = 5/3 = 1.66...$ The constitutive equation (1.5) is often called *polytropic*.

For many liquids, (1.5) is a good approximation, but usually γ is much higher than for a gas, of the order of 10.

Linear deviations from equilibrium. Now we consider small deviations from the state of equilibrium. We write

$$p = p_0 + p' \tag{1.6}$$

$$\rho = \rho_0 + \rho' \tag{1.7}$$

$\mathbf{v}$ is also small, but we don't write a prime on it. The equilibrium state of the fluid is assumed to be homogeneous, which means that it does not depend on $\mathbf{r}$ and t. In the equilibrium state thus the pressure p_0, the density ρ_0 and the entropy s_0 are constants.

In the equations of continuity and the Euler equations we only keep terms linear in the small quantities. As an example,

$$\boldsymbol{\nabla} \cdot (\rho \mathbf{v}) = \rho \boldsymbol{\nabla} \cdot \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \rho \approx \rho_0 = \rho \boldsymbol{\nabla} \cdot \mathbf{v} + \mathbf{v} \cdot \boldsymbol{\nabla} \rho' \approx \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v}$$

This way we obtain the linearized equations

$$\frac{\partial \rho'}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \tag{1.8}$$

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p'. \tag{1.9}$$

We now assume that the velocity field has a potential, this is usually the case when viscosity can be neglected.

$$\mathbf{v} = \boldsymbol{\nabla} \varphi. \tag{1.10}$$

If we introduce the velocity potential into the linearized Euler equation (1.9) we obtain

$$\boldsymbol{\nabla}(\boldsymbol{\rho}_0 \frac{\partial \varphi}{\partial t} + p') = \mathbf{0}$$

This means that $\partial \varphi / \partial t + p'$ has to be a function of t only. But we can add any function of t to φ without changing $\mathbf{v}$. We do that such that

$$p' = -\rho_0 \frac{\partial \varphi}{\partial t} \tag{1.11}$$

This is in fact the linearized Bernoulli equation.

Now, the linearized equation of continuity remains. We have

$$p(\rho_0 + \rho') = p_0 + (\frac{\mathrm{d}p}{\mathrm{d}\rho})_0 \rho' + ..., \tag{1.12}$$

so that

$$p' = c_0^2 \rho', \tag{1.13}$$

where (note that we have tacitly assumed that the derivative is positive; we shall come back to this question later)

$$c_0^2 = \frac{\partial p}{\partial \rho}(\rho_0, s_0). \tag{1.14}$$

For the ideal gas with constant specific heats,

$$c_0^2 = \gamma \frac{p_0}{\rho_0}. \tag{1.15}$$

Now we substitute the linearized Bernoulli equation (1.11) and the expression (1.13) for p' in terms of ρ' into the linearized equation of continuity (1.8). We obtain

$$\boldsymbol{\nabla}^2\varphi - \frac{1}{c_0^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{1.16}$$

This equation is called *d'Alembert's* (1717-1783) *equation* or the *wave equation.* It is clear from (1.13) and (1.10) that the same equation is satisfied by ρ' and p'.

In the sequel, we shall often write ρ, p instead of ρ', p'. So the density and pressure are then $\rho_0 + \rho', p_0 + p$.

Plane waves and stability. As the coefficients in the equation are constants, we first of all look for plane wave solutions of the form $T(t)exp(i\mathbf{k}\cdot\mathbf{r})$. We find

$$\frac{1}{c_0^2}T'' + k^2 T = 0$$

We see that c_0^2 has to be positive or there will be an exponentially growing solution. In other words, the state of equilibrium would be unstable, which is impossible. So we conclude that for stability reasons

$$\frac{\mathrm{d}p}{\mathrm{d}\rho} \geq 0.$$

We then obtain two solutions

$$exp[i(\mathbf{k}\cdot\mathbf{r} \mp kc_0 t)] \tag{1.17}$$

The upper sign gives a wave propagating in the direction of $\mathbf{k}$ and the lower sign one propagating in the opposite direction. The (phase) velocity is c_0.

Let us now take the velocity potential to be one of those solution

$$\varphi = exp[i(\mathbf{k}\cdot\mathbf{r} \mp kc_0 t)]$$

We calculate the velocity field as the gradient of this,

$$\mathbf{v} = i\mathbf{k}exp[i(\mathbf{k}\cdot\mathbf{r} \mp kc_0 t)]$$

It is clear that the velocity vector is parallel to the wave vector $\mathbf{k}$. Sound waves are *longitudinal*.

General solution of 1+1 D wave equation. With one space dimension the wave equation can be rewritten in terms of new variables:

$$\xi = x - c_0 t,\ \eta = x + c_0 t \tag{1.18}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \tag{1.19}$$

$$\frac{\partial}{\partial t} = -c_0\frac{\partial}{\partial \xi} + c_0\frac{\partial}{\partial \eta}. \tag{1.20}$$

Using (1.18-1.20) in the wave equation for the pressure we obtain

$$\frac{\partial^2 p}{\partial \xi \partial \eta} = 0 \tag{1.21}$$

and thus

$$p = f(\xi) + g(\eta) \tag{1.22}$$

or

$$p = f(x - c_0 t) + g(x + c_0 t), \tag{1.23}$$

where f and g are arbitrary functions.

The solution (1.23) is a combination of two waves, one whose shape is described by the function f moving to the right with speed c_0 and the other with shape g moving to the left with speed c_0.

1.1.1. *Solution to the Cauchy problem.* Suppose at time $t = 0$ the pressure $p(x, 0) = F(x)$ and its timederivative $(\partial p/\partial t)(x, 0) = G(x)$ are given. From (1.23) we obtain

$$f(x) + g(x) = F(x), c_0(-f'(x) + g'(x)) = G(x).$$

Hence, (C is a constant)

$$-f(x) + g(x) = \frac{1}{c_0}\int_0^x G(x)\mathrm{d}x + C$$

So that

$$\begin{aligned} f(x) &= \frac{1}{2}(F(x) - \frac{1}{c_0}\int_0^x G(x)\mathrm{d}x - C), \\ g(x) &= \frac{1}{2}(F(x) + \frac{1}{c_0}\int_0^x G(x)\mathrm{d}x + C) \end{aligned}$$

$$p(x,t) = f(x - c_0 t) + g(x + c_0 t) \tag{1.24}$$

$$= \frac{1}{2}(F(x - c_0 t) + F(x + c_0 t)) + \frac{1}{2c_0}\int_{x - c_0 t}^{x + c_0 t} G(x')\mathrm{d}x' \tag{1.25}$$

2. Spherical sound waves

When p is a function of r only it is convenient to write

$$p(r,t) = \frac{R(r,t)}{r} \tag{2.1}$$

In this case

$$\triangle p = \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r} = \frac{1}{r}\frac{\partial^2 R}{\partial r^2}. \tag{2.2}$$

Hence, the wave equation for this case is simply the onedimensional wave equation for R

$$\frac{\partial^2 R}{\partial r^2} - \frac{1}{c_0^2}\frac{\partial^2 R}{\partial t^2} = 0 \tag{2.3}$$

So the general solution is

$$p = \frac{f(r - c_0 t)}{r} + \frac{g(r + c_0 t)}{r}, \tag{2.4}$$

where f and g are arbitrary functions. The first term is an outgoing wave and the second term an ingoing wave.

2.0.2. *The wave equation in presence of fluid source.* For the solution of the wave equation (1.1) appropriate boundary or initial conditions are needed. In many occasions the body acts as a source of fluid. So let us take into account the effect of a source distribution. We assume that the mass of fluid pumped in per volume and time is $\rho_0 q_V$. This means that the mass of fluid in a region is decreasing by outflow and increasing by pumping. So the continuity equation in integral form is then:

$$\frac{\partial}{\partial t}\int \rho \mathrm{d}V = -\int \rho \mathbf{v}\cdot \mathrm{d}\mathbf{S} + \int \rho q_V \mathrm{d}V. \tag{2.5}$$

The differential form of (2.5) is

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = \rho_0 q_V. \tag{2.6}$$

The linearized Bernoulli equation (1.11) is unchanged, but the wave equation now has an extra term. For the pressure it has the form

$$\Delta p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\rho_0 \frac{\partial q_V}{\partial t}. \tag{2.7}$$

3. Energy of acoustic waves

In (2.4) we found that the pressure field in a spherical wave is proportional to $1/r$. To understand this, we are now deriving the energy theorem for acoustical waves. From the linearized Euler system

$$\begin{aligned} \rho_t + \rho_0 \boldsymbol{\nabla}\cdot\mathbf{v} &= 0, \\ \rho_0 \mathbf{v}_{,t} &= -\boldsymbol{\nabla} p \end{aligned}$$

we calculate the time derivative of the kinetic energy of a fixed volume in space. Note that the energy has to be calculated to second order in the small quantities.

$$\begin{aligned} \frac{d}{dt}\int \rho_0 \frac{v^2}{2} dV &= \int \rho_0 \mathbf{v}\cdot\mathbf{v}_{,t} dV \\ &= -\int \mathbf{v}\cdot\boldsymbol{\nabla} p dV = -\int p'\mathbf{v}\cdot d\mathbf{S} + \int p'\boldsymbol{\nabla}\cdot\mathbf{v} dV \\ &= -\int p'\mathbf{v}\cdot d\mathbf{S} - \int \frac{1}{\rho_0} p'\rho_{,t} dV \\ &= -\int p'\mathbf{v}\cdot d\mathbf{S} - \int \frac{c_0^2}{\rho_0}\rho'\rho'_{,t} dV \\ &= -\int p'\mathbf{v}\cdot d\mathbf{S} - \frac{d}{dt}\int \frac{c_0^2}{\rho_0}\frac{\rho'^2}{2} dV \end{aligned}$$

Here we have used $p' = c_0^2\rho'$. Thus

$$\frac{d}{dt}\int \rho_0 [\frac{v^2}{2} + \frac{c_0^2}{\rho_0^2}\frac{\rho'^2}{2}] dV = -\int p'\mathbf{v}\cdot d\mathbf{S}$$

Here, e' is the energy density of the acoustic wave and $\mathbf{j}'_e$ the energy current density

$$\begin{aligned} e' &= \rho_0[\frac{v^2}{2} + \frac{c_0^2}{\rho_0^2}\frac{\rho'^2}{2}] = \rho_0[\frac{v^2}{2} + \frac{p'^2}{2\rho_0^2 c_0^2}], \\ \mathbf{j}'_e &= p'\mathbf{v}. \end{aligned}$$

The same result can be obtained directly from the general expressions for energy in continuum mechanics, see the last section of this chapter.

If we introduce the velocity potential, we have

$$\mathbf{v} = \boldsymbol{\nabla}\varphi, \tag{3.1}$$

$$p' = -\rho_0\frac{\partial\varphi}{\partial t}. \tag{3.2}$$

This gives us

$$e' = \rho_0[\frac{v^2}{2} + \frac{c_0^2}{\rho_0^2}\frac{\rho'^2}{2}] = \rho_0[\frac{(\boldsymbol{\nabla}\varphi)^2}{2} + \frac{(\partial\varphi/\partial t)^2}{2c_0^2}], \tag{3.3}$$

$$\mathbf{j}'_e = p'\mathbf{v} = -\,\rho_0\frac{\partial\varphi}{\partial t}\boldsymbol{\nabla}\varphi. \tag{3.4}$$

If we now consider i spherical acoustic wave, the velocity potential also satisfies the wave equation and thus is of the form (2.4) but with different functions. Let us just kep the outgoing wave

$$\varphi = \frac{F(r - c_0 t)}{r}$$

giving

$$\begin{aligned} p' &= \frac{\rho_0 c_0 F'(r - c_0 t)}{r}, \\ \mathbf{v} &= (\frac{F'(r - c_0 t)}{r} - \frac{F(r - c_0 t)}{r^2})\mathbf{e}_r \end{aligned}$$

As a consequence we obtain

$$\mathbf{j}'_e = (\frac{\rho_0 c_0 (F'(r - c_0 t))^2}{r^2} - \frac{\rho_0 c_0 F'(r - c_0 t)F(r - c_0 t)}{r^3})\mathbf{e}_r$$

So the total energy flux through a sphere of radius r is

$$4\pi(\rho_0 c_0 (F'(r - c_0 t))^2 - \frac{\rho_0 c_0 F'(r - c_0 t)F(r - c_0 t)}{r})$$

The first term remains also when $r \to \infty$ but the second one does not. So terms proportional to $1/r$ in the fields are the ones giving radiation.

4. One frequency. Helmholtz' equation

Waves of a given frequency ω have the form

$$p(\mathbf{x}, t) = p_\omega(\mathbf{x})e^{-i\omega t}. \tag{4.1}$$

The resulting equation for $p_\omega(\mathbf{x})$ is the Helmholtz equation

$$(\Delta + k^2)p_\omega = 0, \tag{4.2}$$

where the wavenumber $k = \omega/c_0$. We note that for the frequency $\omega = 0$ Helmholtz' equation reduces to Laplace's equation

$$\Delta p_\omega = 0. \tag{4.3}$$

Let us first of all look for spherically symmetric solutions of the Helmholtz equation. Just like we did for the full wave equation in the spherically symmetric case, we write

$$p_\omega(r) = \frac{R_\omega(r)}{r}.$$

The resulting equation for R_ω is

$$R''_\omega + k^2 R_\omega = 0$$

with the solutions

$$R_\omega = e^{\pm ikr}$$

and this gives

$$p_\omega = \frac{e^{\pm ikr}}{r}$$

$$p = \frac{e^{i(-\omega t \pm kr)}}{r} \tag{4.4}$$

where the upper sign gives an outgoing spherical wave and the lower sign an ingoing spherical wave compare (2.4). Both waves have a given frequency.

We also note that the outgoing wave for $r \to \infty$ satisfies

$$p_\omega = 0(\frac{1}{r}), \tag{4.5}$$

$$\frac{\partial p_\omega}{\partial r} = ikp_\omega + o(\frac{1}{r}). \tag{4.6}$$

For the ingoing wave there is instead a plussign in front of ik in the second formula here.

4.1. Green's function for Helmholtz' equation. For any function $\phi(\mathbf{x})$ we have

$$\int \delta(\mathbf{x}' - \mathbf{x})\phi(\mathbf{x}')\mathrm{d}V' = \phi(\mathbf{x}). \tag{4.7}$$

When we plug in a function we get the same function back, so what we have here is a unit operator acting of the function.

This makes it interesting to look for a solution of the following equation

$$\boldsymbol{\nabla}^2 g_\omega(\mathbf{r}) + k^2 g_\omega(\mathbf{r}) = -\delta(\mathbf{r}), \tag{4.8}$$

As the source term on the right hand side is spherically symmetric, the solution is also spherically symmetric. Except at the origin, the equation here is the Helmholtz equation. But we have already in (4.4) found the spherically symmetric solutions of the Helmholtz equation. Of these, we have to choose the outgoing solution. We will come back to this later. So we have

$$g_\omega(r) = A\frac{\exp(ikr)}{r},$$

where A is a constant. In order to find the value of the constant A, we integrate (4.8) over a sphere with radius a with center at the origin. On the right hand side we obtain gives -1 from the δ-function. We now let the radius a go to zero. Using $\mathrm{d}V = 4\pi r^2 \mathrm{d}r$ it is easy to see that the integral of the term $k^2 g_\omega$ tends to zero like a^2. In the remaining integral we transform the volume integral by Gauss' theorem

$$\int \Delta g_\omega \mathrm{d}V = \int \boldsymbol{\nabla} g_\omega \cdot \mathrm{d}\mathbf{S} = \frac{\partial g_\omega}{\partial r} 4\pi a^2 \tag{4.9}$$

Calculating the derivative we find that this terms has the limit value $-4\pi A$. We conclude that $-4\pi A = -1$, so that $A = 1/4\pi$.

$$g_\omega(r) = \frac{\exp(ikr)}{4\pi r}, \tag{4.10}$$

The solution (4.10) is called Green's function for infinite space. It satisfies the boundary conditions at infinity that it falls off like $1/r$ and it also satisfies (4.6).

If we instead start by a delta-function at the point $\mathbf{x}_0$ the corresponding Green's function is simply $g_\omega(|\mathbf{x} - \mathbf{x_0}|)$

4.2. Solving the inhomogeneous Helmholtz equation. Often the body generating the sound can be modeled as a source distribution in the Helmholtz equation, which gives the inhomogeneous Helmholtz equation

$$\Delta p_\omega + k^2 p_\omega = -f_\omega(\mathbf{r}), \tag{4.11}$$

We shall now use the Green's function to find the solution. We already know that any function can be written as a superposition of delta-functions, see (4.7). The equation is linear, so we can directly write down the solution as a superposition of Green's functions. The solution is then

$$p_\omega(\mathbf{x}) = \int f_\omega(\mathbf{x}_0) g_\omega(|\mathbf{x} - \mathbf{x}_0|) \mathrm{d}V_0 = \int f_\omega(\mathbf{x}_0) \frac{e^{ik|\mathbf{x} - \mathbf{x_0}|}}{4\pi|\mathbf{x} - \mathbf{x_0}|} \mathrm{d}V_0. \tag{4.12}$$

A physical interpretation of the solution is that the spherical waves from all elementary waves with strength $f_\omega(\mathbf{x}_0)\mathrm{d}V_0$ are added to give the resulting field in the point $\mathbf{x}$.

4.2.1. *Boundary conditions and uniqueness.* But can we be sure that this is the only solution? To answer that question we have to be more careful. The equation (4.11) has to be supplemented with appropriate boundary conditions. We already encountered the problem of uniqueness when we looked for the Green's function. We found that there was one outgoing Green's function and one ingoing. We picked the outgoing one, which is the physically interesting Green's function. We also found that the outgoing Green's function goes to zero as $1/r$ and that its radial derivative is ik times the Green's function with an error that goes to zero faster than $1/r$, see (4.6). Those two conditions pick out the correct Green's function.

Let us take a look at the solution we have found in (4.12). We assume that the source is bounded and choose the origin to lie in the source region. What does the field look like at distances large compared to the size of the source, that is where $r >> r_0$? We consider

$$\begin{aligned} |\mathbf{r} - \mathbf{r}_0| &= \sqrt{\mathbf{r}^2 - 2\mathbf{r}\cdot\mathbf{r}_0 + \mathbf{r}_0^2} = r\sqrt{1 - 2\frac{\mathbf{r}}{r^2}\cdot\mathbf{r}_0 + (\frac{r_0}{r})^2} \\ &= r[1 - \frac{\mathbf{r}}{r^2}\cdot\mathbf{r}_0 + O((\frac{r_0}{r})^2)] = r - \mathbf{e}_r\cdot\mathbf{r}_0 + O(\frac{r_0^2}{r}) \end{aligned}$$

$$p_\omega(\mathbf{x}) \sim \frac{\exp(ikr)}{4\pi r} \int e^{-ik\mathbf{e}_r\cdot\mathbf{r}_0} f_\omega(\mathbf{r}_0) \mathrm{d}V_0 = \frac{\exp(ikr)}{4\pi r} F(k\mathbf{e}_r), \tag{4.13}$$

The function $F(k\mathbf{e}_r)$ only depends on the direction and not on r. We see that the solution we have found for $r \to \infty$ satisfies

$$p_\omega = 0(\frac{1}{r}), \tag{4.14}$$

$$\frac{\partial p_\omega}{\partial r} = ikp_\omega + o(\frac{1}{r}). \tag{4.15}$$

These are called *Sommerfeld's radiation conditions*. We will now show that these conditions uniquely pick out the right solution.

Let us now assume that besides (4.12) we have another solution of the inhomogeneous Helmholtz equation. Let us then denote the difference between the two solutions by p_ω. So p_ω satisfies the homogeneous Helmholtz equation (4.2) and the Sommerfeld radiation conditions (4.15). We multiply (4.10) by p_ω and (4.2) by Green's function g_ω and subtract the equations from each other

$$g_\omega(|\mathbf{r}-\mathbf{r}_0|)\Delta p_\omega(\mathbf{r}) - p_\omega(\mathbf{r})\Delta g_\omega(|\mathbf{r}-\mathbf{r}_0|) = p_\omega(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}_0) \tag{4.16}$$

In this formula we interchange $\mathbf{r}$ and $\mathbf{r}_0$. Integration over $\mathbf{r}_0$ then gives:

$$\begin{aligned} &\int [g_\omega(|\mathbf{r}-\mathbf{r}_0|)\Delta_0 p_\omega(\mathbf{r}_0) - p_\omega(\mathbf{r}_0)\Delta_0 g_\omega(|\mathbf{r}-\mathbf{r}_0|)]\mathrm{d}V_0 \\ = &\int p_\omega(\mathbf{r}_0)\delta(\mathbf{r}-\mathbf{r}_0)\mathrm{d}V_0. \end{aligned} \tag{4.17}$$

The quantity within the parenthesis in the integrand at the lefthand side of this equation is $\boldsymbol{\nabla}_0 \cdot [g_\omega \boldsymbol{\nabla}_0 p_\omega - p_\omega \boldsymbol{\nabla}_0 g_\omega]$. Thus the volume integral can be changed into a surface integral by Gauss' theorem. If the normal component of $\boldsymbol{\nabla}_0$ in the outward direction from the surface is written $\frac{\partial}{\partial n_0}$, then we obtain

$$p_\omega(\mathbf{r}) = \int [g_\omega(|\mathbf{r}-\mathbf{r}_0|)\frac{\partial p_\omega(\mathbf{r}_0)}{\partial n_0} - p_\omega(\mathbf{r}_0)\frac{\partial g_\omega}{\partial n_0}(|\mathbf{r}-\mathbf{r}_0|)]\mathrm{d}S_0.$$

The integration surface here is a large sphere, so $\partial/\partial n_0 = \mathbf{n}\cdot\boldsymbol{\nabla}_0 = \partial/\partial r_0$. On this sphere we know that p_ω as well as the Green's function satisfies the Sommerfeld radiation conditions. The integrand is thus $o(1/r^2)$. So when the radius of the sphere goes to infinity, the surface integral vanishes. We conclude that p_ω vanishes identically. This means that the only solution to the inhomogeneous Helmholtz equation with the Sommerfeld radiation conditions is the one we have already obtained, (4.12).

4.3. Half space Green's function. We have seen that in many cases the sound is generated by sources. They were represented by the function f_ω in the inhomogeneous Helmholtz equation, which was solved for the full space. But in many occasions the sound is instead generated at the boundary of a region. A simple example is that of a plate that is vibrating. This is a crude model of a loudspeaker. We can model this as a halfspace problem. We take the plate to be the plane $z = 0$. The region of the fluid is $z > 0$. The wall is supposed to vibrate so that its velocity is in the normal direction and is a given function $V(x, y)\exp(-i\omega t)$. But the linearized Euler equation (1.9) gives

$$-i\omega\rho_0\mathbf{v}_\omega = -\boldsymbol{\nabla} p_\omega. \tag{4.18}$$

And thus on $z = 0$

$$\frac{\partial p_\omega}{\partial z} = i\omega\rho_0 V. \tag{4.19}$$

So now we need to solve the homogeneous Helmholtz equation in the halfspace $z > 0$ with this boundary condition on the plane $z = 0$. For large r we have the Sommerfeld raditation conditions.

To solve this problem we need a new Green's function. The source is at an arbitrary point $\mathbf{r}_0 = (x_0, y_0, z_0)$ in the region $z_0 > 0$. The Green's function $\tilde{g}_\omega$ solves the equation

$$\nabla^2 \tilde{g}_\omega + k^2 \tilde{g}_\omega = -\delta(\mathbf{r} - \mathbf{r}_0). \tag{4.20}$$

So $\tilde{g}_\omega = \tilde{g}_\omega(\mathbf{r}, \mathbf{r}_0)$. It turns out that $\tilde{g}_\omega$ has to satisfy the same kind of boundary condition as the solution p_ω, but with a zero instead of V. In other words on $z = 0$ it has to satisfy

$$\frac{\partial \tilde{g}_\omega}{\partial z} = 0 \tag{4.21}$$

The full space Green's function $g_\omega(|\mathbf{r} - \mathbf{r_0}|)$ does not satisfy this boundary condition. But if we add a mirror source at the point $\mathbf{r_1} = (x_0, y_0, -z_0)$ we have

$$\tilde{g}_\omega(\mathbf{r}, \mathbf{r}_0) = \frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{4\pi|\mathbf{r} - \mathbf{r_0}|} + \frac{e^{ik|\mathbf{r}-\mathbf{r_1}|}}{4\pi|\mathbf{r} - \mathbf{r_1}|} = \tilde{g}_\omega(\mathbf{r}_0, \mathbf{r}) \tag{4.22}$$

Here,

$$\begin{aligned} |\mathbf{r} - \mathbf{r_0}| &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} \\ |\mathbf{r} - \mathbf{r_1}| &= \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2} \end{aligned}$$

When the source point $\mathbf{r_0}$ is kept fixed, $\tilde{g}$ has the same value in $-z$ as in z so that its derivative with respect to z vanishes at $z = 0$. Note that when the source point is on the plane $z_0 = 0$, $\mathbf{r_1} = \mathbf{r_0}$, so that

$$\tilde{g}_\omega(\mathbf{r}, \mathbf{r}_0) = 2g_\omega(|\mathbf{r} - \mathbf{r_0}|) = \frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{2\pi|\mathbf{r} - \mathbf{r_0}|}$$

So the Green's function is the field from a source at $\mathbf{r}_0$ when the wall is not vibrating.

4.3.1. *The sound field from a vibrating wall.* So we need to find the solution p_ω to the Helmholtz equation (4.2) with the boundary condition (4.19) on the plane and the Sommerfeld radiation conditions at infinity. We now proceed in the same way as we did when we treated the Helmholtz equation in full space. So we multiply (4.2) with the Green's function $\tilde{g}_\omega(\mathbf{r}, \mathbf{r}_0)$ and we take the (4.20) and multiply by p_ω and subtract it. As a result we obtain (4.16) and from it (4.12). The difference now is that the part of the surface integral on the plane $z = 0$ does not vanish. But according to (4.21) one of the terms in the integral over the plane vanishes and we obtain

$$p_\omega(\mathbf{r}) = \int\int \tilde{g}_\omega(\mathbf{r}, \mathbf{r}_0) \frac{\partial p_\omega(x_0, y_0, 0)}{\partial z_0} \mathrm{d}x_0 \mathrm{d}y_0 = \frac{i\omega\rho_0}{2\pi} \int\int \frac{e^{ik|\mathbf{r}-\mathbf{r_0}|}}{|\mathbf{r} - \mathbf{r_0}|} V(x_0, y_0, 0) \mathrm{d}x_0 \mathrm{d}y_0.$$

The solution is of the same kind as the solution of the inhomogeneous Helmholtz equation (4.12). We see that the pressure field is generated by a source distribution on the plane

$$f_\omega = 2\frac{\partial p_\omega(x_0, y_0, 0)}{\partial z_0} \delta(z_0).$$

5. Green's function for the wave equation

We now continue with finding the Green's function of the time-dependent wave equation for boundary conditions at infinity. The equation to be solved is:

$$(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2})G(r,t) = -\delta(\mathbf{r})\delta(t). \tag{5.1}$$

We take the Fourier transform of the equation, i.e. multply by $\exp(i\omega t)$ and integrate with respect to t. We denote the Fourier transform of $G(r,t)$ g_ωOn the right hand side we obtain $-\delta(\mathbf{r})$. As, further, $\partial/\partial t \to i\omega$.

$$g_\omega = \int e^{i\omega t} G(r,t)\mathrm{d}t.$$

The equation then is transformed into

$$\boldsymbol{\nabla}^2 g_\omega(\mathbf{r}) + k^2 g_\omega(\mathbf{r}) = -\delta(\mathbf{r}). \tag{5.2}$$

In other words, the Fourier transform of the Green's function of the wave equation is simply the Green's function of the Helmholtz equation. What remains for us is to transform back to find G. In g_ω we have to write $k = \omega/c_0$

$$G(r,t) = \frac{1}{2\pi}\int e^{-i\omega t} g_\omega(r)\mathrm{d}\omega = \frac{1}{4\pi r}\int e^{i\omega(\frac{r}{c_0}-t)}\mathrm{d}\omega.$$

But

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(i\omega t)\mathrm{d}t = \delta(t) = \delta(-t) \tag{5.3}$$

So we obtain the Green's function

$$G(\mathbf{r},\mathbf{t}) = \frac{\delta(t-\frac{r}{c_0})}{4\pi r}. \tag{5.4}$$

If instead the source is at $\mathbf{r}_0, t_0$ we substitute $\mathbf{r}, t \to \mathbf{r}-\mathbf{r}_0, t-t_0$ we have

$$(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2})G(|\mathbf{r}-\mathbf{r}_0|, t-t_0) = -\delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0). \tag{5.5}$$

The Green's function can now be used for solving the inhomogeneous wave equation with a source term $-s(\mathbf{r},t)$ on the righthand side:

$$(\Delta - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2})p(\mathbf{r},t) = -s(\mathbf{r},t).$$

Identically we have

$$s(\mathbf{r},t) = \int\int s(\mathbf{r}_0,t_0)\delta(\mathbf{r}-\mathbf{r}_0)\delta(t-t_0)\mathrm{d}V_0\mathrm{d}t_0. \tag{5.6}$$

This means that the solution is given by a superposition of Green's functions

$$p(\mathbf{r},t) = \int\int G(\mathbf{r},t;\mathbf{r}_0,t_0)s(\mathbf{r}_0,t_0)\mathrm{d}V_0\mathrm{d}t_0, \tag{5.7}$$

where we have assumed that $s(\mathbf{r}_0,t_0)$ is different from zero within a limited region in space and time.

$$p(\mathbf{r},t) = \int \frac{s(\mathbf{r}_0, t-|\mathbf{r}-\mathbf{r}_0|/c_0)}{|\mathbf{r}-\mathbf{r}_0|}\mathrm{d}V_0. \tag{5.8}$$

We see that the field at the point $\mathbf{r}$ at the time t depends on the source strength at $\mathbf{r}_0$ at the earlier time $t-|\mathbf{r}-\mathbf{r}_0|/c_0$, the time difference being the time necessary for the effect to propagate from $\mathbf{r}_0$ to $\mathbf{r}$. From electromagnetic theory the expression (5.8) is known as a *retarded potential*.

5.1. Green's function for the wave equation in the $2+1$ dimensional case. We know the Green's function for the wave equation. It is

$$G(r,t) = \frac{\delta(t - r/c)}{4\pi r}$$

We are now looking for the Green's function for the case of two space dimensions. We then have to solve

$$\frac{\partial^2 G_2}{\partial x^2} + \frac{\partial^2 G_2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 G_2}{\partial t^2} = -\delta(x)\delta(y)\delta(t).$$

But

$$\delta(x)\delta(y) = \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)\mathrm{d}z.$$

So we can find G_2 by superposition as

$$G_2 = \frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{\delta(t - \sqrt{x^2+y^2+z^2}/c)}{\sqrt{x^2+y^2+z^2}}$$

5.1.1. *A formula for the δ-function.* Now we need to use the following property of a δ function

$$\delta(f(z)) = \sum_n \frac{1}{|f'(z_n)|}\delta(z - z_n)$$

Here, z_n are the zeros of f. It is quite clear that only the zeros of f will contribute. To find the factor in front, let us for simplicity assume that f has just one zero $z_1$and is monotonic and $f(\pm\infty) = \pm\infty$. We introduce the new variable of integration $\zeta = f(z)$, so that $\mathrm{d}\zeta = \mathrm{f}'\mathrm{d}\mathrm{z}'$

$$\int_{-\infty}^{\infty} \delta(f(z))\mathrm{d}z = \int_{-\infty}^{\infty} \delta(\zeta)\frac{1}{f'}d\zeta = \frac{1}{f'(z_1)} = \int_{-\infty}^{\infty} \frac{1}{f'(z)}\delta(z - z_1)dz.$$

If f is decreasing instead, we can introduce $\zeta = |f(z)|$ as a new variable of integration.

5.1.2. *The twodimensional Green's function.* It is clear that $\delta(t - \sqrt{x^2+y^2+z^2}/c)$ is nonzero only for $t \geq r/c$ and that the z-values for which it is nonzero are

$$z_\pm = \sqrt{(ct)^2 - r^2}, r = \sqrt{x^2+y^2}$$

Further,

$$\frac{\partial}{\partial z}(t - \sqrt{x^2+y^2+z^2}/c) = -\frac{z}{c\sqrt{x^2+y^2+z^2}}$$

so that ($H(t)$ is a Heaviside function, which vanishes for negative t and is positive for positive t)

$$\frac{\delta(t - \sqrt{x^2+y^2+z^2}/c)}{\sqrt{x^2+y^2+z^2}} = \frac{H(t - r/c)}{\sqrt{(t^2 - r^2/c^2}}[\delta(z - z_+) + \delta(z - z_-)].$$

We conclude that

FIGURE 1. The slender body is moving to the right with speed V.

$$G_2(\mathbf{r}, t) = \frac{H(t - r/c)}{2\pi\sqrt{t^2 - r^2/c^2}}. \tag{5.9}$$

G_2 is nonzero also in the full interior of the future signal cone but G is nonzero just on the cone. In particular, when $t >> r/c$ we have

$$G_2 \approx \frac{1}{2\pi t}.$$

So the signal has an infinite tail. Close to the moment when the signal arrives to r so that $0 \le t - r/c << r/c$, we have

$$G_2 \approx \frac{\sqrt{c}}{2\pi\sqrt{2}} \frac{1}{\sqrt{r}\sqrt{t - r/c}}.$$

6. Sound from a supersonic projectile

We now consider a slender axially symmetric body moving at supersonic speed. We choose the x-axis along the axis of the body. The velocity is in the positive x-direction and of magnitude V. The cross section of the body is A_B, which then is a function of x, t.

6.1. The body as a source. Consider a small element of length dx. The volume taken up by the part of the body between x and $x + \mathrm{dx}$ is $A_B(x, t)\mathrm{dx}$. In dt the volume increases by $\partial A_B/\partial t \mathrm{dxdt}$. This volume of fluid is thus pumped out from the axis, or per length and time the volume $\partial A_B/\partial t$. Hence the slim body acts like a source on the axis with volume distribution

$$\frac{\partial A_B}{\partial t}\delta(y)\delta(z)$$

(which when integrated over a cylinder between x_1 and x_2 gives $\int_{x_1}^{x_2}(\partial A_B/\partial t)\mathrm{dx}$) Let us now assume that the front of the body is at $x = 0$ at $t = 0$. Let us call the distance from the front along the body ξ, so that $A_B = A_B(\xi)$. On the body we have

$$\xi = Vt - x$$

This means that $A_B = A_B(Vt - x)$ What we have to solve then is the equation

$$\triangle p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = -\rho V^2 A_B''(Vt - x)\delta(y)\delta(z). \tag{6.1}$$

Here we have used that $\partial^2 A_B/\partial t^2 = V^2 A_B''(Vt - x))$. It was consistent to take the density to be constant when calculating the source term, as it is a small term. Another approach leading to the same pressure field would be to use the wave equation in the exterior of the body and apply appropriate boundary conditions on the body.

6.2. Comoving coordinates. A new formal time. The natural thing to do now is to transform to a frame moving with the projectile. We are thus looking for a solution that depends on x and t only in the combination $x_1 = x - Vt$. We then obtain

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = (1 - \frac{V^2}{c^2})\frac{\partial^2}{\partial x_1^2}.$$

When $V < c$ this term has the same sign as the other second space derivatives in the wave operator. But when $V > c$ it has the opposite sign. Hence, more insight is gained introducing a formal time coordinate $t_1 = t - x/V = -x_1/V$ which gives

$$\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = -(\frac{1}{c^2} - \frac{1}{V^2})\frac{\partial^2}{\partial t_1^2} = -\frac{1}{c_1^2}\frac{\partial^2}{\partial t_1^2},$$

where

$$c_1 = \frac{c}{\sqrt{1 - c^2/V^2}}. \tag{6.2}$$

The new equation to be solved is

$$\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{1}{c_1^2}\frac{\partial^2}{\partial t_1^2} = -\rho V^2 A_B''(Vt_1)\delta(y)\delta(z).$$

This is then the wave equation in two space dimensions y, z and one time dimension t_1.

In the source term we can write (for a body of length L, $A_B(\xi)$ vanishes for $\xi > L$)

$$A_B''(Vt_1) = \int_0^\infty A_B''(Vt_1')\delta(t_1' - t_1)\mathrm{d}t_1'.$$

So the source is a superposition of sources along the positive t_1-axis and the solution is a superposition of Green's functions for the $2 + 1$ dimensional case.

$$M \equiv \frac{V}{c_0} \tag{6.3}$$

is called the Mach number. This reformulation of the supersonic wave problem is called *von Karman's acoustical analogy.*

Now we have the solution to our problem as

$$p = \frac{\rho V^2}{2\pi}\int_0^\infty \frac{H(t_1 - t_1' - r/c_1)A_B''(Vt_1')\mathrm{d}t_1'}{\sqrt{(t_1 - t_1')^2 - r^2/c_1^2}}.$$

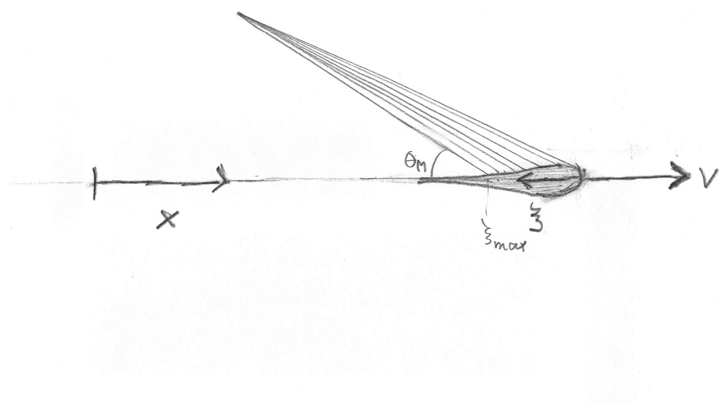


FIGURE 2. The signals reaching a point of observation. The Mach cone is defined by the last signal.

So changing the variable of integration we obtain

$$p = \frac{\rho V^2}{2\pi} \int_0^{max(0,Vt-x-r\sqrt{M^2-1})} \frac{A_B''(\xi)\mathrm{d}\xi}{\sqrt{(Vt-x-\xi)^2-(M^2-1)r^2}}.$$

6.2.1. *The Mach cone.* The field is nonzero in the region

$$\xi_m = Vt - x - r\sqrt{M^2-1} \geq 0$$

This is a cone with apex at the front of the body and opening angle $2\theta_M$, where

$$tan\theta_M = \frac{1}{\sqrt{M^2-1}}, sin\theta_M = \frac{1}{M}.$$

So ξ_m denotes from how far back on the body signals reach the point x, r at time t.

$$\xi_m = Vt - x - r\sqrt{M^2-1} = Vt - M(xsin\theta_M + r\cos\theta_M) = (t - \frac{\mathbf{n}\cdot\mathbf{r}}{c})V$$

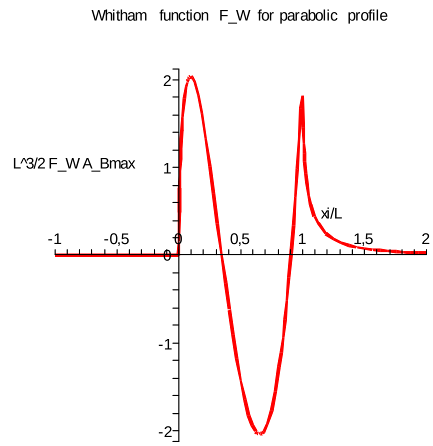
Here, $\mathbf{n}$ is the normal to the cone.

$$\mathbf{n} = \sin\boldsymbol{\theta}_{\mathbf{M}}\mathbf{e}_{\mathbf{x}} + \cos\theta_M\mathbf{e}_r.$$

6.3. Asymptotic result for large r. Let us denote the length of the body by L. We recall that $0 \leq \xi, \xi_m \leq L$. We now consider the asymptotic region where $\sqrt{M^2-1}r >> L$.

We write

$$\begin{aligned} & (Vt-x-\xi)^2 - (M^2-1)r^2 \\ = & [(Vt-x-r\sqrt{M^2-1}) - \xi][(Vt-x+r\sqrt{M^2-1}) - \xi] \\ = & [\xi_m - \xi][2r\sqrt{M^2-1} + (\xi_m - \xi)] \approx [\xi_m - \xi]2r\sqrt{M^2-1} \end{aligned}$$



This gives us

$$p \approx \frac{\rho V^2}{2^{3/2}\pi(M^2-1)^{1/4}}\frac{1}{\sqrt{r}}\int_0^{max(0,\xi_m)}\frac{A_B''(\xi)\mathrm{d}\xi}{\sqrt{\xi_m-\xi}} \tag{6.4}$$

$$\frac{\rho V^2}{(M^2-1)^{1/4}\sqrt{2r}}F_W(V(t-\frac{\mathbf{n}\cdot\mathbf{r}}{c})).$$

Here we have $\xi_m = Vt - x - r\sqrt{M^2-1}$. Whitham's function F_W is defined as

$$F_W(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\xi}\frac{A_B''(\mu)\mathrm{d}\mu}{\sqrt{\xi-\mu}}$$

$$= \frac{1}{2\pi}\int_0^{\infty}\frac{A_B''(\xi-\eta)\mathrm{d}\eta}{\sqrt{\eta}} = \frac{1}{2\pi}\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\int_0^{\infty}\frac{A_B(\xi-\eta)\mathrm{d}\eta}{\sqrt{\eta}}. \tag{6.5}$$

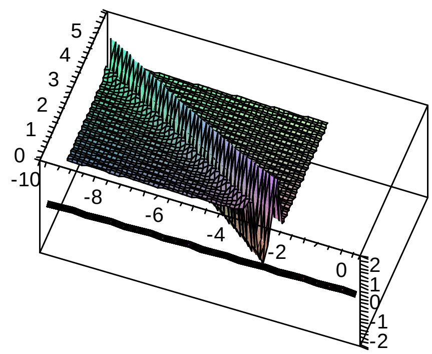
The integral here depends only on the geometry of the sound source. It is interesting to note from (6.4) that the pressure decays as $r^{-1/2}$ for constant $t - \mathbf{n}\cdot\mathbf{r}/c$. From the linearized Euler equation we have

$$-\rho_0 i\omega\mathbf{v} = -\boldsymbol{\nabla} p.$$

This shows that the dominating term in the velocity also decays as $r^{-1/2}$ for constant $t - \mathbf{n}\cdot\mathbf{r}/c$. Hence the energy current density will be proportional to r^{-1}, see (3.4). This means that the energy flux through a large cylinder will be independent of r, whichs is characteristic of radiation. We also note that $F_W(\xi) = 0$ for $\xi < 0$. We assume that $A_B(\xi)$ approaches zero faster than $\xi^{3/2}$ for $\xi \to 0$. Then $F_W(\xi)$ is finite for $\xi = 0$.

In the plot is shown the Whitham function for a body generated by a parabola rotating around the axis.The maximum radius is in the middle of the body. The corresponding maximum area is denoted by $A_{B\max}$ The following picture shows the pressure distribution in the asymptotic region. The thick line is the body. The horisontal coordinates are $(x-ct)/L, r/L$ with ranges $[-10, 0]$ and $[0, 5]$ and the vertical coordinate is the pressure

*Whitham's asymptotic expression for the solution. M=2



7. Acoustic energy

The linearized equations are (here we write p' and ρ' for the perturbations)

$$\begin{aligned} \rho'_{,t} + \rho_0 \boldsymbol{\nabla} \cdot \mathbf{v} &= 0, \\ \rho_0 \mathbf{v}_{,t} &= -\boldsymbol{\nabla} p'. \end{aligned}$$

We consider a fixed region in space and calculate the rate of change of the kinetic energy

$$\begin{aligned} \frac{\partial}{\partial t} \int \frac{v^2}{2} \rho_0 dv &= \int \mathbf{v} \cdot \rho_0 \mathbf{v}_{,t} dv = - \int \mathbf{v} \cdot \boldsymbol{\nabla} p' dv \\ &= - \int \mathbf{v} \cdot \boldsymbol{\nabla} p' dv + \int p' \boldsymbol{\nabla} \cdot \mathbf{v} dv \end{aligned}$$

In the first term here we use Gauss' theorem. In the second term we use $p' = c^2 \rho'$ and then the equation of continuity to find

$$p' \boldsymbol{\nabla} \cdot \mathbf{v} = c^2 \rho' \boldsymbol{\nabla} \cdot \mathbf{v} = -\frac{c^2}{\rho_0} \rho' \rho'_{,t} = -\frac{\partial}{\partial t} (\frac{c^2}{2\rho_0} \rho'^2)$$

The result is

$$\frac{\partial}{\partial t} \int (\frac{v^2}{2} \rho_0 + \frac{c^2}{2\rho_0} \rho'^2) dv = - \int p' v \cdot d\mathbf{s}$$

The integral on the left hand side is the acoustic energy

$$e_a = \frac{v^2}{2} \rho_0 + \frac{c^2}{2\rho_0} \rho'^2$$

and

$$\mathbf{j}_a = p' \mathbf{v}$$

is the acoustic energy current density.

Let us also for completeness derive the same result directly from continuum mechanics. The elastic energy (per mass) is given by

$$(7.1)\qquad \begin{aligned} d\varepsilon &= -pd(\frac{1}{\rho}), \\ \varepsilon &= \int_{\rho_0}^{\rho} \frac{p}{\rho^2} d\rho + \varepsilon_0. \end{aligned}$$

Differentiating, we find

$$\begin{aligned} \frac{d\varepsilon}{d\rho} &= \frac{p}{\rho^2}, \\ \frac{d^2\varepsilon}{d\rho^2} &= -\frac{2p}{\rho^3} + \frac{1}{\rho^2}\frac{dp}{d\rho}. \end{aligned}$$

Hence, up to second order

$$(7.2)\qquad \varepsilon = \varepsilon_0 + \frac{p_0}{\rho_0^2}\rho' + \frac{1}{2}[-\frac{2p_0}{\rho_0^3} + \frac{1}{\rho_0^2}c_0^2]\rho'^2 + ...$$

The energy per volume is given by

$$(7.3)\qquad e = \rho(\frac{v^2}{2} + \varepsilon).$$

Hence

$$(7.4)\qquad e = \rho_0\varepsilon_0 + (\frac{p_0}{\rho_0} + \varepsilon_0)\rho' + \frac{1}{2}[\rho_0 v^2 + \frac{c_0^2}{\rho_0}\rho'^2] + ...$$

Let us rewrite it as

$$e = -p_0 + (\frac{p_0}{\rho_0} + \varepsilon_0)\rho + \frac{1}{2}[\rho_0 v^2 + \frac{c_0^2}{\rho_0}\rho'^2] + ...$$

The energy current is

$$\begin{aligned} \mathbf{j}_e &= e\mathbf{v}+p\mathbf{v} = (-p_0 + (\frac{p_0}{\rho_0} + \varepsilon_0)\rho + p)\mathbf{v} + ... \\ &= [(\frac{p_0}{\rho_0} + \varepsilon_0)\rho + p']\mathbf{v} + ... \end{aligned}$$

The energy equation is

$$(7.5)\qquad e_{,t} + \boldsymbol{\nabla}\cdot\mathbf{j}_e = 0.$$

The constant term in e gives no contribution. Further, we have a term

$$(\frac{p_0}{\rho_0} + \varepsilon_0)[\rho_{,t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v})]$$

which vanishes because of mass conservation. What remains then is

$$(7.6)\qquad e_{a,t} + \boldsymbol{\nabla}\cdot\mathbf{j}_a = 0$$

Here,

$$e_a = \rho_0[\frac{v^2}{2} + \frac{c_{0^2}}{2\rho_0^2}\rho'^2],$$
$$\mathbf{j}_a = p'\mathbf{v}.$$

This is the same result as we obtained earlier directly from the linearized Euler system. What we have done is that we have subtracted the tranport of energy caused by the constant background state.

CHAPTER 2

Linear dispersive waves

1. Dispersion. Water waves

It is wellknown that the sunlight, by means of a prism, can be decomposed in its components, from the red light to the violet light. This phenomenon is due to the fact that light of different frequency is differently refracted. The most refracted light has the lowest velocity. The velocity v is a decreasing function $v(\nu)$ of the frequency ν and thus the violet light, which has the highest frequency, is the most refracted light. Instead of the frequency ν we will use the angular frequency $\omega = 2\pi\nu$ and instead of v we will use the wave number $k = \frac{2\pi}{\lambda}$, where λ is the wavelength. The three quantities v, ν and λ have the relation $v = \nu\lambda$, so anyone of them can be studied as a function of anyone of the two others. We will study the function $\nu(\frac{1}{\lambda})$, or, with our notation chosen, the function $\omega(k)$. This functional dependence is called the *dispersion relation*.

The notion of *group velocity* will be illustrated by means of the superposition of two harmonic waves with the same amplitude, frequencies ω_1, ω_2 and wave numbers k_1, k_2:

$$u(x,t) = A\cos(k_1x - \omega_1 t) + A\cos(k_2x - \omega_2 t), \tag{1.1}$$

where ω_1, ω_2, k_1, k_2 are all positive and $k_2 > k_1$. Equation (1.1) is rewritten:

$$u(x,t) = 2A\cos(\frac{k_2-k_1}{2}x - \frac{\omega_2-\omega_1}{2}t)\cos(\frac{k_1+k_2}{2}x - \frac{\omega_1+\omega_2}{2}t). \tag{1.2}$$

The equation (1.2) describes a harmonic wave with wave number $\frac{1}{2}(k_1+k_2)$, frequency $\frac{1}{2}(\omega_1+\omega_2)$ and consequently wave velocity $\frac{\omega_1+\omega_2}{k_1+k_2}$. This wave is modulated by another wave with wave number $\frac{1}{2}(k_2-k_1)$, frequency $\frac{1}{2}(\omega_2-\omega_1)$ and propagation velocity $\frac{\omega_2-\omega_1}{k_2-k_1}$. If we assume that $k_2 - k_1 << k_1$ and thus $\omega_2 - \omega_1 << \omega_1$, then the velocity of the wave $V(k)$ and the velocity of the modulation $U(k)$ can be written

$$V(k) = \frac{\omega(k)}{k} \tag{1.3}$$

$$U(k) = \frac{d\omega}{dk}. \tag{1.4}$$

The velocity $V(k)$ is called *phase velocity* and the velocity $U(k)$ is called *group velocity*. The group velocity is the propagation velocity of a group of waves limited by the modulation. In many applications the group velocity is the effective velocity for propagation of signals and transport of energy.

We obtain the same result if we consider a superposition of plane waves

$$\phi(x,t) = \int_{-\infty}^{\infty} F(k)\exp(i(kx - \omega(k)t))dk. \tag{1.5}$$

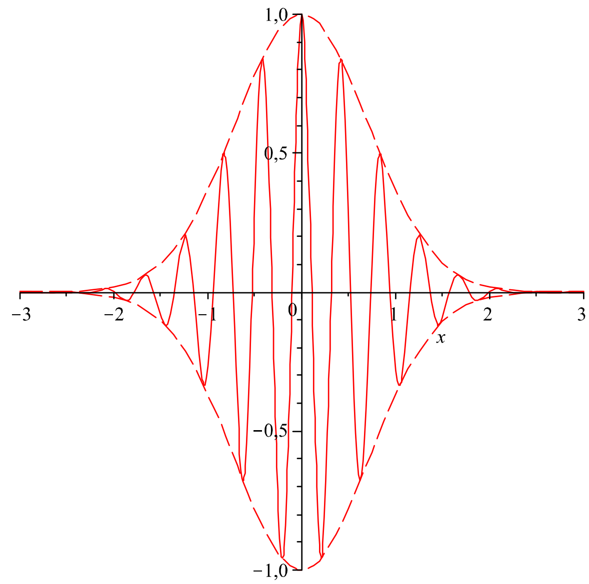


FIGURE 1. Wave packet

We assume that $F(k)$ is peaked around $k = k_0$ and is narrow enough that $\omega(k)$ can be represented as a straight line for the wave numbers where $F(k)$ is different from zero.Putting $l = k - k_0$ we can write

$$\omega(k) \approx \omega(k_0) + \frac{d\omega}{dk}(k_0)l.$$

We then obtain

$$\phi(x,t) \approx \exp(i(k_0 x - \omega(k_0)t))G(x - \frac{d\omega}{dk}(k_0)t). \tag{1.6}$$

Here

$$G(\xi) = \int_{-\infty}^{\infty} F(k_0 + l)\exp(i\xi l)dl$$

In the figure we have drawn the real part of $\phi(x,t)$. The functions $\pm G(x - \frac{d\omega}{dk}(k_0)t)$ are dashed.

Sound waves in fluids have practically the same velocity independent of their frequency. This means that the dispersion relation for sound waves is

$$\omega = c_0 k \tag{1.7}$$

and this case is called no dispersion. Dispersion occurs when (1.7) is replaced by a nonlinear dependence $\omega(k)$. As we will see, this is the case for water waves.

As an example of dispersion and group velocity we will study the linear theory of water waves. A nondissipative, incompressible, homogeneous fluid (water) is placed in a constant gravitational field. The spatial coordinates are (x, y, z) and the corresponding fluid velocity components are $(v_x, v_y, v_z = v)$. The gravitational force is directed in the negative z direction. By putting the fluid density ρ constant in the continuity equation (1.2) and adding $-\rho g\mathbf{e}_z$ to the force term in Navier-Stokes' equations (1.3) in Chapter 1 we obtain the fundamental equations for the fluid motion:

$$\nabla \cdot \mathbf{v} = 0 \tag{1.8}$$

$$\rho(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\nabla p - \rho g\mathbf{e}_z. \tag{1.9}$$

We consider irrotational motion, which means

$$\mathbf{v} = \nabla\phi. \tag{1.10}$$

Using the vector identity

$$\frac{1}{2}\nabla(\mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v}) \tag{1.11}$$

and the consequence of (1.10)

$$\nabla \times \mathbf{v} = 0, \tag{1.12}$$

we can integrate (1.9) with the result

$$\frac{p - p_0}{\rho} = -\phi_t - \frac{1}{2}(\nabla\phi)^2 - gz + B(t), \tag{1.13}$$

where $B(t)$ is an arbitrary function and p_0 is an arbitrary constant, for reasons of convenience separated from $B(t)$. The function $B(t)$ can be absorbed in the velocity potential by making the change

$$\phi \to \phi - \int B(t)dt, \tag{1.14}$$

so that in practice the term $B(t)$ can be dropped in (1.13). The equations (1.8) and (1.10) give Laplace's equation for ϕ:

$$\Delta\phi = 0. \tag{1.15}$$

The problem thus is to solve Laplace's equation with appropriate boundary conditions. Then the physically interesting quantities p and $\vec{v}$ are obtained from (1.13) and (1.10) respectively. The boundary conditions must be discussed first.

The fluid is contained in a vessel with air above it. We can describe the upper fluid surface using the equation

$$z = \zeta(x, y, t), \tag{1.16}$$

We have

$$\frac{D\zeta}{Dt} \equiv \frac{\partial\zeta}{\partial t} + v_x\frac{\partial\zeta}{\partial x} + v_y\frac{\partial\zeta}{\partial y} = v_z = \frac{\partial\phi}{\partial z}. \tag{1.17}$$

The equation (1.17) is a kinematical condition on the fluid surface. There is also a dynamical condition: the forces in the media on both sides of the boundary surface must be equal. If the surface tension in the fluid is neglected, this condition means that the pressure on both sides of the boundary surface must be equal. At the surface we therefore have the condition:

$$p = p_0, \tag{1.18}$$

where p is the pressure in the fluid, given by (1.13) and p_0 is the constant valöue of the external air pressure. We thus neglect the effect of the motion of the fluid on the air pressure just above the boundary surface. Using (1.13) and (1.16) we can express the boundary condition (1.18) in terms of ϕ and ζ. The same thing can be done with the boundary condition (1.17) using (1.10). For the boundary conditions we then obtain:

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \zeta}{\partial y} = \frac{\partial \phi}{\partial z} \tag{1.19}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + gz = 0. \tag{1.20}$$

The reason why two boundary conditions are needed instead of one is that the boundary surface $z = \zeta(x, y, t)$ is unknown.

Now we assume that the fluid velocity and the height of the boundary above the equilibrium surface are small quantities. If $z = 0$ means the equilibrium of the fluid surface, then the quadratical terms in the small quantities ϕ and ζ can be neglected in (1.19) and (1.20). We then obtain the linear boundary conditions

$$\zeta_t = \phi_z \tag{1.21}$$

$$\phi_t + g\zeta = 0. \tag{1.22}$$

The boundary conditions (1.21), (1.22) are, like (1.19) and (1.20), valid for $z = \zeta(x, y, t)$ and not for $\zeta = 0$. However, this difference gives a second order effect and is neglected in (1.21) and (1.22). Therefore we apply the boundary conditions (1.21), (1.22) for $z = 0$. Eliminating ζ we obtain

$$\phi_{tt} + g\phi_z = 0,\ z = 0. \tag{1.23}$$

The equation of the bottom surface is

$$z = -h_0(x, y). \tag{1.24}$$

The kinematical condition that the normal component of the fluid velocity is zero at the bottom surface gives a boundary condition analogous to (1.19). We just replace ζ by $-h_0$, use the fact that the time derivative of h_0 is zero and obtain:

$$\frac{\partial \phi}{\partial x}\frac{\partial h_0}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial h_0}{\partial y} + \frac{\partial \phi}{\partial z} = 0,\ z = -h_0(x, y). \tag{1.25}$$

For a horizontal bottom surface the boundary condition (1.25) becomes

$$\frac{\partial \phi}{\partial z} = 0,\ z = -h_0. \tag{1.26}$$

The boundary value problem for water waves in the linear approximation and with a horizontal bottom surface is summarized below, using the equations (1.15), (1.23) and (1.26):

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0,\ -h_0 < z < 0 \tag{1.27}$$

$$\phi_{tt} + g\phi_z = 0,\ z = 0 \tag{1.28}$$

$$\phi_z = 0,\ z = -h_0. \tag{1.29}$$

After solving (1.27) - (1.29) and having obtained ϕ, the equation for the water surface is obtained from (1.22):

$$z = \zeta(x, y, t) = -\frac{1}{g}\phi_t(x, y, 0, t). \tag{1.30}$$

We will now study solutions to (1.27) - (1.29) of the form

$$\phi = Z(z)\exp(i(k_x x + k_y y) - i\omega t) \tag{1.31}$$

$$\zeta = A\exp(i(k_x x + k_y y) - i\omega t). \tag{1.32}$$

Solutions of the form (1.31), (1.32) mean wave propagation in the (x, y)-plane. Inserting (1.31) into (1.27) we find the condition

$$Z'' - k^2 Z = 0, \tag{1.33}$$

where $k^2 = k_x^2 + k_y^2$. The equation (1.33) gives together with (1.29)

$$Z = Z(-h_0)\cosh(k(z + h_0)). \tag{1.34}$$

The condition (1.30) gives, by insertion of (1.31) and (1.32):

$$A = \frac{i\omega}{g} Z(0). \tag{1.35}$$

By (1.34) and (1.35) we find

$$Z = \frac{g}{i\omega} A \frac{\cosh(k(z + h_0))}{\cosh(kh_0)}. \tag{1.36}$$

The solution for ζ and ϕ then becomes

$$\zeta = A\exp(i(k_x x + k_y y) - i\omega t) \tag{1.37}$$

$$\phi = -\frac{ig}{\omega} A \frac{\cosh(k(z + h_0))}{\cosh(kh_0)} \exp(i(k_x x + k_y y) - i\omega t). \tag{1.38}$$

The boundary condition (1.28) is not yet used. Inserting (1.38) into (1.28) we obtain:

$$-\omega^2 \cosh(kh_0) + gk\sinh(kh_0) = 0 \tag{1.39}$$

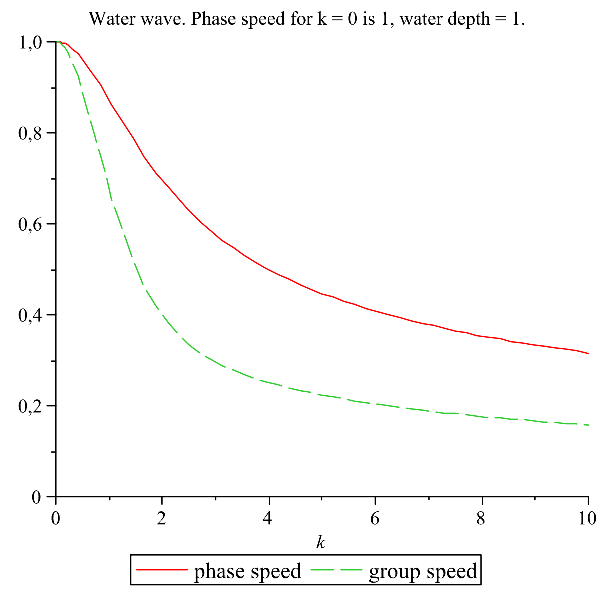
or

$$\omega^2 = gk\tanh(kh_0), \tag{1.40}$$

which is the dispersion relation for linear water waves, propagating on the surface of a fluid with constant equilibrium height h_0 above the bottom surface. For the depoendence of ω on k we introduce the function $W(k)$. In the case of linear water waves we thus have

$$W(k) = \sqrt{(gk\tanh(kh_0))},\ \omega = \pm W(k). \tag{1.41}$$

It is convenient to choose W as an odd function of k ($\tanh x/x$ is a non-negative function of x)



$$W(k) = \sqrt{gh_0}k\sqrt{\tanh(kh_0)/(kh_0)}. \tag{1.42}$$

2. The stationary phase method

The linear water wave problem is an example of a boundary value problem for a differential equation, whose solution, in one space dimension, has the form

$$\phi(x,t) = A\exp(i(kx - W(k)t)), \tag{2.1}$$

where $W(k)$ is a given function of k. In the water wave problem new solutions can be obtained by superposition of solutions of the form (2.1) with different k values. The most general superposition is given by a Fourier integral

$$\phi(x,t) = \int_{-\infty}^{\infty} F(k)\exp(i(kx - W(k)t))dk. \tag{2.2}$$

A method will now be developed for evaluating the integral (2.2) asymptotically for $t \to \infty$ with $\frac{x}{t}$ constant. Physically this means that we study the group of waves represented by the integral in (2.2), when the group has propagated a long distance with constant velocity (the group velocity). The integral in (2.2) is now written:

$$\phi(x,t) = \int_{-\infty}^{\infty} F(k)\exp(-i\chi(k)t)dk, \tag{2.3}$$

where

$$\chi(k) = W(k) - k\frac{x}{t}. \tag{2.4}$$

In the equation (2.3) $\chi(k)$ is considered as a function of k with $\frac{x}{t}$ as a fixed parameter. For great t values the main contribution to the integral in (2.3) must come from a region in the neighborhood of the point k_0, where χ is stationary. For k values far from the stationary point the integral in (2.3) is expected to oscillate so fast so that the contributions to the integral cancel. The inventor of this "stationary phase method" was Lord Kelvin (1824-1907), and henceforth we will follow his argumentation.

The functions $F(k)$ and $\chi(k)$ in (2.3) are expanded in Taylor series in the neighborhood of $k = k_0$:

$$F(k) \approx F(k_0) \tag{2.5}$$

$$\chi(k) \approx \chi(k_0) + \frac{1}{2}(k-k_0)^2\chi''(k_0). \tag{2.6}$$

We assume that $\chi''(k_0) \neq 0$. With the approximations (2.5), (2.6) inserted into (2.3) we obtain

$$\phi(x,t) = F(k_0)\exp(-i\chi(k_0)t)\int_{-\infty}^{\infty}\exp\{-\frac{i}{2}(k-k_0)^2\chi''(k_0)t\; dk. \tag{2.7}$$

The integral here is the wellknown. It converges for $\mathrm{Re}(\alpha) \geq 0$ or $-\pi/2 \leq \arg(\alpha) \leq \pi/2$. We introduce $\zeta = \sqrt{\alpha}z$. Here the root is picked with $-\pi/4 \leq \arg(\sqrt{\alpha}) \leq \pi/4$. The integration over ζ is then pushed to the real line.

$$\int_{-\infty}^{\infty}\exp(-\alpha z^2)dz = \frac{1}{\sqrt{\alpha}}\int \exp(-\zeta^2)d\zeta = \sqrt{\frac{\pi}{\alpha}}. \tag{2.8}$$

Writing (sgn stands for the sign)

$$\chi''(k_0) = |\chi''(k_0)|\mathrm{sgn}(\chi''(k_0))$$

$$\alpha = \frac{i}{2}\chi''(k_0)t = \frac{1}{2}\exp(i\frac{\pi}{2}\mathrm{sgn}\chi''(k_0))|\chi''(k_0)t|, \tag{2.9}$$

$$\sqrt{\alpha} = \sqrt{\frac{|\chi''(k_0)t|}{2}}\exp(i\frac{\pi}{4}\mathrm{sgn}\chi''(k_0)) \tag{2.10}$$

We conclude that

$$\phi(x,t) = F(k_0)\sqrt{\frac{2\pi}{t|\chi''(k_0)|}}\exp\{-i\chi(k_0)t - i\frac{\pi}{4}\mathrm{sgn}\chi''(k_0)\}. \tag{2.11}$$

In (2.11) we have found an approximation of the Fourier integral (2.3). The approximation becomes better when t increases. That is why that part of the interval $-\infty < k < \infty$, where (2.5), (2.6) is a bad approximation, becomes less important when t increases.

3. A Fourier method for the linear water wave problem

The velocity potential $\phi(x,y,z,t)$ and the surface $z = \zeta(x,y,t)$ was determined by Laplace's equation (1.27) with the boundary conditions (1.28), (1.29) to give the result (2.5), (2.6). In order to complete the determination we also need an initial condition. We will make this determination in this section with the simplification that we consider one-dimensional water waves, which means that ϕ and ζ do not depend on y.

The general solution for $\zeta(x,t)$ is a sum of solutions of the type (1.37) for all possible values of k, which means a Fourier integral

$$\zeta(x,t) = \int_{-\infty}^{\infty}F(k)\exp(i(kx-\omega t))dk \tag{3.1}$$

We have two solutions, called modes, $\omega = \pm W(k)$ to the dispersion relation (1.40) with $W(k)$ given in (1.41). A superposition of the two modes gives:

$$\zeta(x,t) = \int_{-\infty}^{\infty} F_1(k)\exp[i(kx - W(k)t)]dk + \int_{-\infty}^{\infty} F_2(k)\exp[i(kx + W(k)t)]dk. \tag{3.2}$$

As $W(k) > 0$ for $k > 0$ and is an odd function of k, see (1.41) the first term on the righthand side of (3.2) means waves propagating in the right direction (unchanged phase for growing x and t) and the second term means waves propagating in the left direction (unchanged phase for decreasing x and increasing t).

We now assume the initial conditions on the water surface:

$$\zeta(x,0) = \zeta_0(x) \tag{3.3}$$

$$\zeta_t(x,0) = 0. \tag{3.4}$$

With the condition (3.4) we obtain from (3.2):

$$F_1(k) = F_2(k) = F(k). \tag{3.5}$$

The condition (3.3) then gives with (3.2) and (3.5):

$$\zeta_0(x) = 2\int_{-\infty}^{\infty} F(k)\exp(ikx)dk \tag{3.6}$$

with the inverse Fourier transform

$$F(k) = \frac{1}{4\pi}\int_{-\infty}^{\infty} \zeta_0(x)\exp(-ikx)dx. \tag{3.7}$$

From (1.41) $W(k)$ seems to have a singularity for $k = 0$. This singularity can be removed; if we choose the behaviour $k\sqrt(gh)$ near $k = 0$, then $W(k)$ is unique and analytic on the whole k-axis. Then $W(k)$ becomes an odd function, which implies that the first term in $\zeta(x,t)$ means waves propagating to the right and the second term means waves propagating to the left. The stationary phase method now gives an expression for the first term in $\zeta(x,t)$ in (3.2). To the equation

$$W'(k) - \frac{x}{t} = 0 \tag{3.8}$$

we have, with $W(k)$ given by (1.41), two solutions $k = \pm k_0$. From (2.11) we now obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} F(k)\exp(i(kx - W(k))dk \\ = \; & F(k_0)\sqrt(\frac{2\pi}{t|W''(k_0)|})\exp\{ik_0x - iW(k_0)t - i\frac{\pi}{4}\text{sgn}W''(k_0)\} \\ & +F(-k_0)\sqrt(\frac{2\pi}{t|W''(-k_0)|})\exp\{-ik_0x - iW(-k_0)t - i\frac{\pi}{4}\text{sgn}W''(-k_0)\}. \end{aligned} \tag{3.9}$$

From (3.7) we obtain immediately

$$F(-k_0) = F^*(k_0), \tag{3.10}$$

where F^* is the complex conjugated of F. We will show below that $W''(k_0) < 0$ for $k_0 > 0$. As $W(k)$ is an odd function, so is its second derivitive. This means that the two terms on the righthand side of (3.9) are complex conjugated to each other and their sum, which we call $\zeta^r(x,t)$ (the superscript "r" means right-going waves), can be written:

$$\zeta^r(x,t) = 2\mathrm{Re}\{F(k)\sqrt{(}\frac{2\pi}{t|W''(k)|})\exp(ikx - iW(k)t + \frac{\pi i}{4})\},\ t\to\infty,\ \frac{x}{t} > 0, \tag{3.11}$$

where k is now the positive root of the equation (3.8).

Let us now for simplicity introduce dimensionless variables $k^* = kh_0$, $\omega^* = \sqrt{h_0/g}\omega$. We then have According to (1.3) and (1.4) the phase velocity $c(k)$ and the group velocity $C(k)$ are given as:

$$W^*(k^*) = \sqrt{k^*\tanh k^*}, \tag{3.12}$$

$$c^*(k^*) = \frac{W^*}{k^*} = \sqrt{\frac{\tanh(k^*)}{k^*}} \tag{3.13}$$

We first show that the phase speed is strictly decreasing with the wave number. We find, using $\cosh^2 k^* - \sinh^2 k^* = 1$ and $\sinh 2k^* = 2\sinh k^*\cosh k^*$

$$\frac{d}{dk^*}c^{*2} = \frac{k^* - \frac{1}{2}\sinh 2k^*}{k^{*2}\cosh^2 k^*}.$$

But $\sinh 2k^*/k^*$ is a strictly increasing function for $k^* > 0$. Hence, the phase velocity $c(k)$ is a strictly decreasing function. Let us now consider the group velocity

$$\begin{aligned} C^* &= \frac{dW^*}{dk^*} = \frac{1}{2W^*}(\tanh k^* + \frac{k^*}{\cosh^2 k^*}) \\ &= \frac{c^*}{2}(1 + \frac{2k^*}{\sinh 2k^*}) \end{aligned} \tag{3.14}$$

Obviously C is a strictly decreasing function of k. This is also seen clearly in the figure (1).

For a finite depth h_0 the maximal group velocity $\sqrt{(}gh_0)$ is attained for $kh_0 \to 0$. The velocity $\sqrt{(}gh_0)$ thus is the velocity of the foremost point of disturbation, where the waves with the longest wavelength occur. The front of the disturbation if followed by waves with successively decreasing wavelength. On the other hand, for $h_0 k \to 0$ the expression (3.11) for the wave can no longer be used, because $W''(k)$ tends to zero when k tends to zero.

4. Kinematic derivation of the group velocity

The waveform given in (3.11) for rightgoing waves is obtained by an approximate summation of the superposition (3.1) of elementary waves for great t values and $\frac{x}{t} > 0$. Since k in (3.11) is the positive root of the equation (3.8), k becomes a function of x and t. Thus the procedure of summation of elementary waves gives us a wave of the form

$$\zeta(x,t) = \mathrm{Re}\{A(x,t)\exp(i\theta(x,t))\}, \tag{4.1}$$

whose amplitude and phase both depend on space and time. The amplitude $A(x,t)$ and the phase $\theta(x,t)$ are obtained from (3.8) as:

$$A(x,t) = 2\exp(\frac{\pi i}{4})F(k(x,t))\sqrt{(}\frac{2\pi}{t|W''(k(x,t))|}) \tag{4.2}$$

$$\theta(x,t) = k(x,t)x - W(k(x,t))t. \tag{4.3}$$

The form of the nonuniform wave (4.1) is the same as of the monochromatic wave (2.1), but neither the amplitude nor the distance in space and time between two successive wave maxima are constant.

For the nonuniform wave we introduce the concepts of wave number and frequency by defining them as θ_x and $-\theta_t$ respectively. In the nonuniform case we cannot, of course, obtain any well-defined quantities by counting the number of wave maxima per unit time or unit length. For our new definitions of wave number and frequency we obtain using (4.3):

$$\frac{\partial\theta}{\partial x} = k + (x - W'(k))\frac{\partial k}{\partial x} \tag{4.4}$$

$$\frac{\partial\theta}{\partial t} = -W(k) + (x - W'(k)t)\frac{\partial k}{\partial t}. \tag{4.5}$$

The condition (3.8) eliminates the terms with k_x and k_t in (4.4) and (4.5) and we obtain

$$\frac{\partial\theta}{\partial x} = k(x,t) \tag{4.6}$$

$$\frac{\partial\theta}{\partial t} = W(k(x,t)) = \omega(x,t). \tag{4.7}$$

The wave number k, which was first introduced as a certain value of the wave number in the Fourier integral (3.1), where it was called k_0, thus agrees with the definition of the local wave number θ_x in a nonuniform oscillating wave. Furthermore, the local wave number and the local frequency satisfy the dispersion relation even in the nonuniform case.

The intuitive interpretation of θ_x as a wave number is meaningful only if θ_x does not change too much during *one* oscillation. From (3.8) we obtain

$$W''(k)k_x = \frac{1}{t} = \frac{W'}{x} \tag{4.8}$$

and thus

$$\frac{k_x}{k} = \frac{W'}{kW''}\frac{1}{x} \tag{4.9}$$

and analogously

$$\frac{k_t}{k} = -\frac{W'}{kW''}\frac{1}{t}. \tag{4.10}$$

If the distance x contains a great number of wavelengths, it follows from (4.9) that the relative change of k in a wavelength is small. The same conclusion can be made for the relative change of k in a period, if t contains a great number of periods. Thus $k(x,t)$ and $\omega(x,t)$, for x and t great in the abovementioned meaning, fulfil the condition of being slowly varying functions of x and t.

According to (3.8) a definite value k_0 of the function $k(x,t)$ is found on the straight line

$$x = W'(k_0)t. \tag{4.11}$$

This means that an observer travelling with the velocity $W'(k_0)$ always sees waves with the wave number k_0 and the frequency $W(k_0)$. The quantity

$$W'(k) = \frac{d\omega}{dk} \tag{4.12}$$

is the *group velocity*, introduced in a way different from (1.4). In a superposition like (2.2) of waves with different wave number and frequency a certain wave number k_0 propagates with the velocity $W'(k_0)$ and moves the distance $W't$ in the time t.

A definite value θ_0 of the phase $\theta(x,t)$ is found in the (x,t)-plane on the curve

$$\theta(x,t) = \theta_0. \tag{4.13}$$

This θ-value propagates according to the equation

$$\theta_x \frac{dx}{dt} + \theta_t = 0. \tag{4.14}$$

Using (4.6) and (4.7) we obtain from (4.14):

$$\frac{dx}{dt} = \frac{\theta_t}{\theta_x} = \frac{\omega}{k}. \tag{4.15}$$

Thus we have arrived at the same definition of phase velocity as in (1.3). The local phase velocity is the velocity of an observer attached to one and the same wave maximum. Such an observer sees how the local wave number and the local frequency are changed. This phenomenon will now be studied in an example.

The motion of a homogeneous beam which is not affected by forces except at its ends, is described by the equation

$$\gamma^2 z_{xxxx} + z_{tt} = 0, \tag{4.16}$$

where γ^2 is a constant:

$$\gamma^2 = \frac{EI}{\lambda}. \tag{4.17}$$

Here E is the modulus of elasticity, I is the surface moment of inertia of the beam intersection with respect to the central line of the beam and λ is the beam mass per unit length. If the beam is in equilibrium, the central line is the x-axis and $z(x,t)$ is the deviation from the x- axis of that point on the central line, whose coordinate is x in equilibrium. Without specifying any boundary conditions we can obtain a dispersion relation from (4.16) by the oscillatory solution

$$z = A \exp(i(kx - \omega t)), \tag{4.18}$$

which, after insertion into (4.16), gives

$$\gamma^2 k^4 - \omega^2 = 0. \tag{4.19}$$

If we solve the equation (4.19) for ω and choose the positive sign we obtain

$$\omega = W(k) = \gamma k^2. \tag{4.20}$$

From (3.8) and (4.20) we obtain

$$W'(k) = 2\gamma k = \frac{x}{t} \tag{4.21}$$

and consequently

$$k = \frac{x}{2\gamma t} \tag{4.22}$$

$$\omega = \frac{x^2}{4\gamma t^2} \tag{4.23}$$

$$\theta = kx - \omega t = \frac{x^2}{4\gamma t}. \tag{4.24}$$

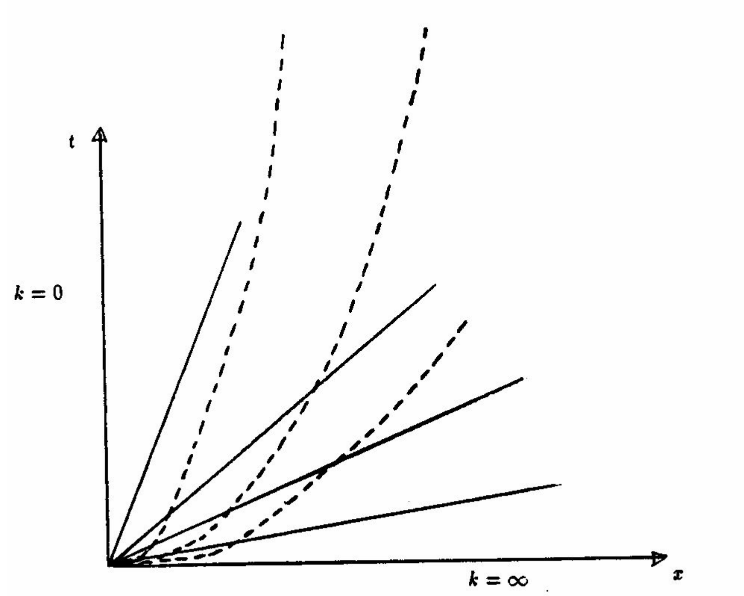


FIGURE 2. Group lines (continuous) and phase lines (dashed) of waves along a homegeneous beam.

An observer travelling with the group velocity moves in the (x, t)-plane along curves with constant $W'(k)$, which means according to (4.21):

$$\frac{x}{t} = \text{constant}. \tag{4.25}$$

An observer travelling with the phase velocity moves in the (x, t)-plane along curves with constant θ, which means according to (4.24):

$$\frac{x^2}{t} = \text{constant} \tag{4.26}$$

The group lines and phase lines are drawn in the (x, t)-plane in Fig. 2.

As is seen from the figure a phase line (4.26) cuts group lines (4.25) with successively decreasing k value and successively increasing wavelength. An observer travelling with the local phase velocity attached to one and the same wave maximum thus sees the wave maxima in his neighborhood at growing distances. On the other hand, an observer travelling with the local group velocity always passes new wave maxima, because the phase velocity $\frac{\omega}{k} = \gamma k$ is half the group velocity $2\gamma k$.

Starting from an arbitrary superposition (3.1) of plane waves with different wave numbers and frequencies we have introduced the concept of group velocity in (3.8) by evaluating this superposition by the method of stationary phase. However, group velocity is such a fundamental concept, that it ought to be introduced without reference to a Fourier integral evaluated by a special method. There are also problems in wave propagation, for example waves in inhomogeneous media, which are not solved by a Fourier integral like (3.1). Even in such cases the concept of group velocity should be possible to introduce.

In order to fulfil the program of introducing group velocity without reference to a Fourier integral we assume the existence of a nonuniform wave with a phase function $\theta(x,t)$. We *define* wave number and frequency as θ_x and $-\theta_t$ respectively. That these definitions are meaningful is obvious from our earlier considerations, leading to (4.4) and (4.5). In this earlier case we also had at our disposal a dispersion relation $\omega = W(k)$ and the relation (3.8), valid for x and t large. Now we start with the equations

$$k = \theta_x,\ \omega = -\theta_t \tag{4.27}$$

and assume a functional dependence

$$\omega = W(k), \tag{4.28}$$

but we do not have the relation (3.8), which was derived as a consequence of a Fourier integral calculated by the stationary phase method. By derivation we eliminate θ from (4.27) and obtain

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0. \tag{4.29}$$

By use of (4.28) we obtain

$$\frac{\partial \omega}{\partial x} = W'(k)\frac{\partial k}{\partial x} \tag{4.30}$$

and after insertion of (4.30) into (4.29)

$$\frac{\partial k}{\partial t} + c(k)\frac{\partial k}{\partial x} = 0, \tag{4.31}$$

where

$$c(k) = W'(k). \tag{4.32}$$

The equation (4.31) is a nonlinear first order partial differential equation. We will solve it by a general method in next chapter. It is remarkable that even though the original problem, from which the dispersion relation (4.28 was derived, is linear, the complete solution of the wave problem implies the nonlinear equation (4.31).

CHAPTER 3

Nonlinear hyperbolic waves propagating in one direction

1. The kinematic wave equation

We consider a second order partial differential operator

$$L \equiv \sum_{i,j=1}^{n} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{1.1}$$

When L operates on $\exp[\Sigma \xi_i x_i]$ this corresponds to the substitution

$$\frac{\partial}{\partial x_i} \rightarrow \xi_i.$$

The differential operator L is replaced by the quadratic form

$$L \rightarrow \widehat{L} \equiv \sum_{i,j=1}^{n} \alpha_{ij} \xi_i \xi_j.$$

This quadratic form can be diagonalized. If the signs of the resulting diagonal form all have the same sign, L is said to be *elliptic*. The most basic example is the Laplace equation- If one term has opposite sign it is said to be *hyperbolic* and the wave equation is a basic example. The wave operator can be factorized.

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \frac{\partial^2}{\partial x^2}\right) = \left(\frac{\partial}{\partial t} - c_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x}\right). \tag{1.2}$$

If only one of the factors is retained in (1.2), just one of the two terms in (1.24) is a solution. If we retain only

$$\left(\frac{\partial}{\partial t} + c_0 \frac{\partial}{\partial x}\right)\rho = 0, \tag{1.3}$$

the general solution is an arbitrary function of $x - c_0 t$:

$$\rho = f(x - c_0 t). \tag{1.4}$$

This is the simplest hyperbolic wave problem and almost trivial. The nonlinear counterpart

$$\rho_t + \tilde{c}(\rho)\rho_x = 0, \tag{1.5}$$

where $\tilde{c}(\rho)$ is a given function of ρ is nontrivial and we will see that a study of this problem, which we have met already in (4.31), leads to the essential ideas for treating nonlinear hyperbolic waves.

The key to the solution of (1.5) lies in the question: What is the condition for the lefthand side of (1.5) to be a total derivative? The total derivative $d\rho(x,t)/dt$ is calculated as

$$\frac{d\rho(x,t)}{dt} = \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial t}. \tag{1.6}$$

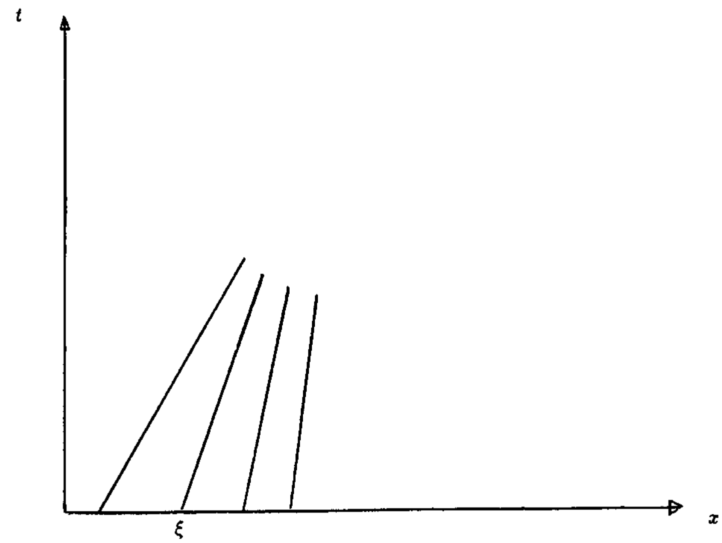


FIGURE 1. Straight lines starting at $x = \xi$ and having slope $dx/dt = \tilde{c}(\rho)$

Comparing (1.5) and (1.6) we now find the answer to the question above: Along every curve in the (x, t)-plane with the slope

$$\frac{dx}{dt} = \tilde{c}(\rho(x,t)) \tag{1.7}$$

the lefthand side of (1.5) is a total derivative. Consider such a curve C. On the curve C we thus have

$$\frac{d\rho}{dt} = 0 \tag{1.8}$$

$$\frac{dx}{dt} = \tilde{c}(\rho). \tag{1.9}$$

According to (1.8) ρ is constant on the curve C. This means according to (1.9) that dx/dt is constant on C. Consequently the curves C are straight lines in the (x, t)- plane. The solution of (1.5) thus depends on the construction of a multitude of straight lines (Fig. 1) in the (x, t)-plane. Each of the straight lines has a slope $\tilde{c}(\rho)$ corresponding to the ρ value belonging to the straight line.

We now consider an initial value problem for the equation (1.5) and assume:

$$\rho(x, t = 0) = f(x). \tag{1.10}$$

If a curve C intersects the x-axis $t = 0$ at $x = \xi$, then $\rho = f(\xi)$ is constant on this curve. The slope of the curve is $\tilde{c}(f(\xi))$, which is called $F(\xi)$; it is a known function. The curve C has the equation

$$x = \xi + F(\xi)t. \tag{1.11}$$

Now the solution of (1.5) with the initial condition (1.10) can be written down; it is simply

$$\rho = f(\xi), \tag{1.12}$$

where ξ is implicitly given as a function of x and t through (1.11).

The solution (1.12) will now be verified. From (1.12) we have

$$\rho_t = f'(\xi)\xi_t \tag{1.13}$$

$$\rho_x = f'(\xi)\xi_x \tag{1.14}$$

The derivatives ξ_t and ξ_x are obtained through implicit derivation of (1.11):

$$0 = \xi_t + F'(\xi)\xi_t t + F(\xi) \tag{1.15}$$

or

$$\xi_t = -\frac{F(\xi)}{1 + F'(\xi)t}. \tag{1.16}$$

In the same way we find

$$\xi_x = \frac{1}{1 + F'(\xi)t}. \tag{1.17}$$

From (1.15)-(1.17) we obtain

$$\rho_t = -\frac{F(\xi)f'(\xi)}{1 + F'(\xi)t} \tag{1.18}$$

$$\rho_x = \frac{f'(\xi)}{1 + F'(\xi)t} \tag{1.19}$$

It is now obvious from (1.18) and (1.19) that (1.5) is satisfied, because $\tilde{c}(\rho) = F(\xi)$.

A glance at the multitude of curves in Fig. 1 directly leads to the question: What happens when the curves intersect? As we know they are associated with different values of ρ. In order to answer the question we return to the equations (1.7)-(1.9). They can be interpreted in the sense that every value of ρ propagates with a velocity $\tilde{c}(\rho)$, specific for the ρ-value considered. If ρ is given as $\rho = f(x)$ at the time $t = 0$, then according to (1.11) and(1.12) a solution at the time t is obtained by moving each point on the curve $\rho = f(x)$ the distance $\tilde{c}(\rho)t = F(\xi)t$ to the right (Fig. 1); this distance is different for different ρ.

As we know, a constant ρ-value is associated with a straight line $x = \xi + \tilde{c}(\rho)t$ in the (x, t)-plane. If $\frac{d\tilde{c}(\rho)}{d\rho} > 0$ the propagation velocity increases with ρ and the front of the wave profile steepens. ρ_x and ρ_t become infinite for

$$t = -\frac{1}{F'(\xi)}. \tag{1.20}$$

A front steepening of the wave corresponds to negative $F'(\xi)$.

What is primarily interesting is where this occurs for the first time. We find this by differentating t with respect ot ξ

$$\frac{\mathrm{d}t}{\mathrm{d}\xi} = \frac{F''(\xi)}{(F'(\xi))^2}. \tag{1.21}$$

So it occurs first where $F''(\xi)$ vanishes and at the same time $F'(\xi) < 0$. In other words for a ξ value for which the initial distribution has a point of inflexion and is decreasing with y.

Eventually, the front steepening gives a profile like that shown in Fig. 2. In Fig. 3 there is an x-interval in which three ρ-values correspond to each x-value. This phenomenon corresponds to intersection of curves C in Fig. 1. If ρ is a physical

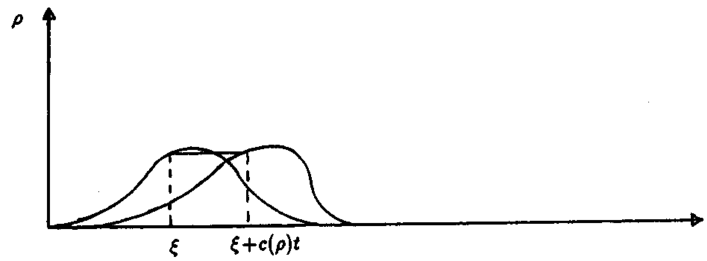


FIGURE 2. Construction of $\rho(x,t)$ (right curve) from $\rho(x,0)$ (left curve)

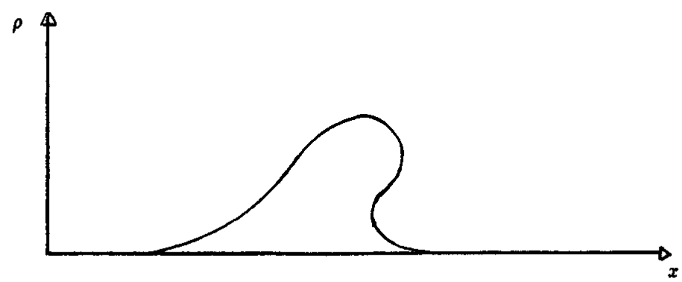


FIGURE 3. Front steepening of wave giving a multivalued wave profile

quantity like density or pressure, this is an unacceptable physical situation and a better physical theory must be formulated. This will be done in this chapter by allowing discontinuities in the solution and giving rules for their handling.

Now we can solve the nonlinear first order partial differential equation (4.31), which describes how the wave number in a nonuniform wave propagates with the group velocity. If

$$k(x, t = 0) = f(x), \tag{1.22}$$

where $f(x)$ is a given function, the solution is

$$k = f(\xi), \tag{1.23}$$

where ξ is given implicitly as a function of x and t through the equation

$$x = \xi + \tilde{c}(\xi)t, \tag{1.24}$$

where

$$\tilde{c}(\xi) = \tilde{c}(f(\xi)). \tag{1.25}$$

If the initial distribution $k = f(x)$ is such that $k \neq 0$ only for $x \approx 0$, then we use $\xi \approx 0$ in (1.24) and obtain a determination of $k(x,t)$:

$$x \approx \tilde{c}(\xi)t = \tilde{c}(f(\xi))t = \tilde{c}(k)t \tag{1.26}$$

and we have reproduced the same derivation of the group velocity as that leading to (3.8). The asymptotic equation (1.26), and consequently (3.8), is valid for x and t so great, that the spatial extension of the initial disturbation can be neglected, and the disturbation at $t = 0$ is thus treated as if it were concentrated to $x = 0$. Thus the exact relation (1.24), in which this approximation is not made, is a progress in comparison with (3.8), which is based on the stationary phase method.

2. Nonlinear sound waves

In Chapter 1 we have seen how the fundamental hydrodynamic equations for an ideal fluid (1.2) and (1.3) and the constitutive equation for an ideal fluid (1.5) give the wellknown linear wave equation. Now we will again start with the same equations but take into account the lowest order deviations from the linearized equations. We restrict our study to waves propagating in one dimension only, so that the variables are functions of x, t only.

The continuity equation then becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{2.1}$$

In nonlinear acoustics it is common to use the letter p for the deviation of pressure from some bacground value p_0. So let us use a capital P for the total pressure. The Euler equations reduce to a single equation:

$$\rho(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}) = -\frac{\partial P}{\partial x}. \tag{2.2}$$

The constitutive equation for the fluid is conveniently written

$$\rho = \rho(P, s), \tag{2.3}$$

where s is the entropy.

In fluids, heat conduction is usually a very slow process. This means that it can be neglected except for waves of very low frequency. Further, viscosity can usually be neglected, except for waves of very high frequency. One can then show from thermodynamics that the entropy is a constant in the motion. We can thus take

$$s = s_0 \tag{2.4}$$

So, the constitutive equation simplifies to

$$\rho = \rho(P, s_0). \tag{2.5}$$

In particular, for an ideal gas, with constant specific heat,

$$P = a\rho^\gamma, \tag{2.6}$$

where $\gamma = c_p/c_v$ and a is a constant, which depends on the value of the constant entropy s_0. For two-atomic gases $\gamma = 7/5 = 1.4$ except for very low and very high temperatures. For monatomic gases, $\gamma = 5/3 = 1.66...$

For many liquids, (2.6) is a good approximation, but usually γ is then much higher, of the order of 10. *give a reference here*

$$\rho(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}) = -c^2\frac{\partial \rho}{\partial x}. \tag{2.7}$$

Ee can consider ρ and c, defined as

$$c \equiv (a\gamma)^{1/2} P^{\frac{\gamma-1}{2\gamma}} = c_0(\frac{P}{p_0})^{\frac{\gamma-1}{2\gamma}} = c_0(1+p)^{\frac{\gamma-1}{2\gamma}}. \tag{2.8}$$

as functions of only the pressure p. For an ideal gas with constant specific heats,

$$c \equiv (\frac{\partial \rho(p,s)}{\partial p})^{-\frac{1}{2}} \tag{2.9}$$

We conclude that we have two equations for two functions v and one of the thermodynamic variables ρ or p.

The two equations (2.1) and (2.7) cannot in general be solved explicitly. There is, however, an important class of exact solutions, which in effect involve one unknown function only. We find them by assuming that v is actually a function of p. This assumption gives for (2.1) and (2.2):

$$\frac{d\rho}{dp}\frac{\partial p}{\partial t} + \frac{d(\rho v)}{dp}\frac{\partial p}{\partial x} = 0 \tag{2.10}$$

$$\rho\frac{dv}{dp}\frac{\partial p}{\partial t} + (\rho v\frac{dv}{dp} + 1)\frac{\partial p}{\partial x} = 0. \tag{2.11}$$

The equations (2.10), (2.11) are a homogeneous system of equations for the two variables $\partial p/\partial t$ and $\partial p/\partial x$. It has nontrivial solutions if and only if its determinant of coefficients vanishes, or

$$\frac{d\rho}{dp}(\rho v\frac{dv}{dp} + 1) - \rho\frac{dv}{dp}\frac{d(\rho v)}{dp} = 0. \tag{2.12}$$

After simplification, we find

$$\frac{d\rho}{dp} - \rho^2(\frac{dv}{dp})^2 = 0. \tag{2.13}$$

We have already in chapter 1 introduced the speed of sound for linear sound waves

$$\frac{d\rho}{dp} = \frac{1}{c^2} \tag{2.14}$$

we obtain from (2.13)

$$\frac{dv}{dp} = \pm\frac{1}{\rho c}, \tag{2.15}$$

To find the class of exact solutions, we assumed v to be a function of p. Here we see that the specific form of this function: if we integrate the above equation, we find

$$v = \pm\int\frac{\mathrm{d}p}{\rho c}. \tag{2.16}$$

Note that as the entropy is constant, ρ and c are unique functions of p.

In particular for an ideal gas with constant specific heats,

$$\int_0^p \frac{\mathrm{d}p}{\rho c} = \frac{2(c-c_0)}{\gamma-1}.$$

which gives, inserted into (2.10) or (2.11):

$$\frac{\partial p}{\partial t} + \{v(p) \pm c(p)\}\frac{\partial p}{\partial x} = 0. \tag{2.17}$$

The equation (2.17) describes a disturbation whose local propagation velocity is $v \pm c$, where v is the local fluid velocity and c is the local sound velocity, which means the local velocity of wave propagation relative to the fluid. The upper sign in (2.17) means waves propagating to the right and the lower sign means waves propagating to the left.

We choose to study waves propagating to the right according to the equation

$$\frac{\partial p}{\partial t} + (v + c)\frac{\partial p}{\partial x} = 0. \tag{2.18}$$

From now on p will mean the deviation from the pressure of the fluid in equilibrium, which thus has $p = 0$. If the acoustic waves have small amplitude, we can replace ρ and c_0 in (2.15) and obtain

$$v = \frac{p}{\rho_0 c_0}. \tag{2.19}$$

Thus p can be replaced by v as the dependent variable in the equation (2.18).

For the derivation of (2.18) only the continuity equation (2.1) and Newton's second law (2.2) have been used together with the existence of a constitutive law (2.3) with constant entropy s_0. In order to be able to solve our nonlinear acoustic wave problem we must assume a specific form of the constitutive equation (2.3). This assumption will give us the desired dependence of c on p.

A two terms Taylor expansion of $c(p)$ gives

$$\begin{aligned} c &= c_0 + (\frac{\partial c(p,s)}{\partial p})_{s=s_0,p=0}p \\ &= c_0 + (\frac{\partial c(p,s)}{\partial p})_{s=s_0,p=0}\rho_0 c_0 v, \end{aligned} \tag{2.20}$$

where c_0 is the wave velocity in the fluid in equilibrium. From (2.20) we obtain

$$c + v = c_0 + \beta v = c_0 + \frac{\beta p}{\rho_0 c_0} = c_0[1 + \frac{\beta(\rho - \rho_0}{\rho_0}], \tag{2.21}$$

where

$$\beta = 1 + \rho_0 c_0(\frac{dc}{dp})_0 = 1 + \frac{1}{2}\rho_0(\frac{dc^2}{dp})_0. \tag{2.22}$$

The subscript "0" after the derivatives in (2.22) means that the derivative is taken at the equilibrium value $p = 0$. Using (2.8) and the fact that only processes with constant entropy $s = s_0$ are considered we rewrite the last derivative in (2.22):

$$\frac{dc^2}{dp} = \frac{d}{dp}(\frac{dp}{d\rho}) = \frac{d^2p}{d\rho^2}\frac{d\rho}{dp} = \frac{\frac{d^2p}{d\rho^2}}{\frac{dp}{d\rho}}. \tag{2.23}$$

A frequently used notation is

$$A = (\rho\frac{dp}{d\rho})_0 \tag{2.24}$$

$$B = (\rho^2\frac{d^2p}{d\rho^2})_0. \tag{2.25}$$

The quantities A and B are used in a Taylor expansion of the excess pressure p as a function of the relative density change $\frac{\Delta\rho}{\rho_0}$ from equilibrium:

$$p = A\frac{\Delta\rho}{\rho_0} + \frac{1}{2}B(\frac{\Delta\rho}{\rho_0})^2. \tag{2.26}$$

The ratio $\frac{B}{A}$ is a measure of the nonlinearity in the adiabatic equation of state, which gives a relation between p and ρ. Using (2.23)-(2.25) in (2.22) we obtain

$$\beta = 1 + \frac{1}{2}\frac{B}{A}. \tag{2.27}$$

The ratio $\frac{B}{A}$ is an important quantity to measure for fluids transmitting acoustical pulses. As an example of a medium, for which $\frac{B}{A}$ can be calculated, we choose the ideal fluid. For this the pressure at constant entropy, according to Poisson's law (2.6), is proportional to ρ^γ, where $\gamma = \frac{c_p}{c_v}$. Consequently using (2.24), (2.25):

$$A \sim (\rho\gamma\rho^{\gamma-1})_0 = (\gamma\rho^\gamma)_0 = \gamma p_0 \tag{2.28}$$

$$B \sim (\rho^2\gamma(\gamma-1)\rho^{\gamma-2})_0 = \gamma(\gamma-1)p_0. \tag{2.29}$$

>From (2.28), (2.29) we find that

$$\frac{B}{A} = \gamma - 1 \tag{2.30}$$

and consequently from (2.27)

$$\beta = \frac{\gamma+1}{2}. \tag{2.31}$$

Using (2.19) and (2.21) we write (2.18) as an equation for v:

$$\frac{\partial v}{\partial t} + (c_0 + \beta v)\frac{\partial v}{\partial x} = 0, \tag{2.32}$$

where the constant β is determined for the ideal fluid in (2.31).

The equation (2.32) can now be solved using the procedure developed for solving (1.5) with an initial condition. The initial condition for (2.32) is chosen as :

$$v(x, t = 0) = f(x). \tag{2.33}$$

The solution (1.11), (1.12) gives for (2.32), (2.33) the solution

$$v = f(\xi), \tag{2.34}$$

where ξ is implicitly given as a function of x and t through the relation

$$x = \xi + [c_0 + \beta f(\xi)]t. \tag{2.35}$$

A solution of (2.32) will now be graphically analyzed for a simple harmonic sound source. This leads to a boundary value problem with the boundary condition at $x = 0$:

$$v(0, t) = v_0 \sin\omega t. \tag{2.36}$$

However, the solution of (2.32) was given in (2.34), (2.35) under the assumption that a condition at $t = 0$ is known. Whether an initial condition or a boundary condition is appropriate depends on the experimental situation. It is obvious that our method of solution of (2.32) with an initial condition can also be used if the initial condition is replaced by a boundary condition. We only change the positions of x and t in the earlier analysis, writing (2.32) as

$$\frac{\partial v}{\partial x} + \frac{1}{c_0 + \beta v}\frac{\partial v}{\partial t} = 0. \tag{2.37}$$

The local propagation velocity $\frac{dx}{dt} = c_0 + \beta v$ is thus replaced by the "slowness" $\frac{dt}{dx} = (c_0 + \beta v)^{-1}$. The solution of (2.37) with (2.36) obtained analogously to the solution to (2.32) with (2.33) is

$$v(x,t) = v_0 \sin \omega\psi, \tag{2.38}$$

where $\psi(x,t)$ is implicitly given by the relation

$$t = \psi + \frac{x}{c_0 + \beta v_0 \sin \omega\psi}. \tag{2.39}$$

By (2.38) and (2.39) the solution can also be written implicitly as an equation to be solved for $v(x,t)$:

$$v(x,t) = v_0 \sin\{\omega(t - \frac{x}{c_0 + \beta v})\}. \tag{2.40}$$

In (2.40) it is practical to use the retarded time

$$\tau = t - \frac{x}{c_0}. \tag{2.41}$$

Using (2.41) in (2.40) we obtain

$$v = v_0 \sin\{\omega(\tau + \frac{x}{c_0}\frac{\beta_0 \frac{v}{c_0}}{1 + \beta\frac{v}{c_0}})\}. \tag{2.42}$$

The pure sine oscillation of the wave at $x = 0$ will be deformed with increasing x-values. We will now study the deformation of the wave profile. Using the Mach number

$$M = \frac{v_0}{c_0} \tag{2.43}$$

we write (2.42):

$$\omega\tau = \arcsin(\frac{v}{v_0}) - \frac{\omega}{c_0}x\frac{\beta M \frac{v}{v_0}}{1 + \beta M \frac{v}{v_0}}. \tag{2.44}$$

We note that the terms containing β are the nonlinear contributions. If we keep only the lowest nonlinear contribution we have the simpler equation

$$\omega\tau = \arcsin(\frac{v}{v_0}) - \sigma\frac{v}{v_0}, \tag{2.45}$$

where

$$\sigma = \frac{\beta}{c_0^2}\omega v_0 x. \tag{2.46}$$

The equation (2.45) is studied graphically in Fig. 4. It is seen that for increasing σ, which means longer distance from the monoharmonic sound source, the negative slope of the straight line increases and the deformation of the wave profile becomes stronger. The deformation of the wave profile thus is a cumulative effect,

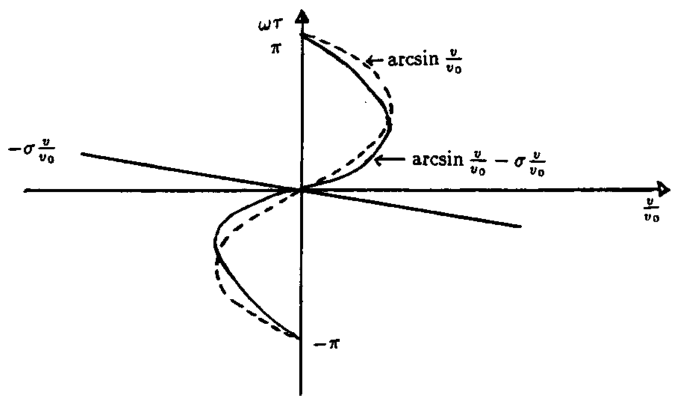


FIGURE 4. Deformed periodical wave (continuous) at nondimensional distance sigma from a boundary with a pure sin oscillation (dashed)

a result of contributions which add to each other during the propagation of the wave. Derivation of (2.45) gives:

$$\frac{d(\omega\tau)}{d(\frac{v}{v_0})} = \frac{1}{\sqrt{(1-\frac{v^2}{v_0^2})}} - \sigma. \tag{2.47}$$

>From (2.47) we find that for $\sigma = 1$ the derivative of $\omega\tau$ with respect to $\frac{v}{v_0}$ becomes zero for $\frac{v}{v_0} = 0$. This means that the derivative of v with respect to $\omega\tau$ becomes infinite. For $\sigma > 1$ the implicit function $v(t)$, given by (2.45), is not unique and cannot be used. As was mentioned in Section 3.1 the nonuniqueness can be removed by allowing discontinuities in the solution.

3. Shock waves

In the nonlinear wave theory, studied in Sections 3.1 and 3.2, we have met the problem of multivalued solutions to the first order partial nonlinear differential equation considered. It is already mentioned that one way of avoiding the dilemma is to introduce discontinuous solutions. However, the differential equation is not valid at the discontinuity. There it must be replaced by an integral equation on an interval covering the discontinuity. The continuity equation (2.1) has the physical meaning of conservation of mass. In our subsequent analysis, with the assumption (2.4) of constant entropy, it follows from (2.3), (2.8) and (2.15) that ρv can be considered as a function of ρ:

$$\rho v = j(\rho). \tag{3.1}$$

Using (3.1) the continuity equation (2.1) can be written

$$\rho_t + j'(\rho)\rho_x = 0. \tag{3.2}$$

The equation (3.2) has the same form as (1.5).

We will now integrate the continuity equation over a volume with intersection area 1 and fixed endpoints x_1 and x_2. The result is

$$\frac{d}{dt}\int_{x_1}^{x_2} \rho dx = (\rho v)_{x_1} - (\rho v)_{x_2}. \tag{3.3}$$

The interpretation of the equation (3.3) is that the increment of mass per unit time in the volume is the difference per unit time between the entering mass in x_1 and the outgoing mass in x_2.

We now assume that ρ has a discontinuity at $x_{sh}(t)$ between x_1 and x_2. The lefthand side of (3.3) is rewritten with two integrals:

$$\begin{aligned}\frac{d}{dt}\int_{x_1}^{x_2} \rho dx &= \frac{d}{dt}\{\int_{x_1}^{x_{sh-}} \rho dx + \int_{x_{sh+}}^{x_2} \rho dx\} \\ &= (\rho_- - \rho_+)\frac{dx_{sh}}{dt} + \int_{x_1}^{x_{sh-}} \frac{\partial \rho}{\partial t} dx + \int_{x_{sh+}}^{x_2} \frac{\partial \rho}{\partial t} dx,\end{aligned} \tag{3.4}$$

where plus and minus correspond to the right and left side of the discontinuity respectively.

For x_1 and x_2 approaching each other the two last integrals in (3.4) vanish and we obtain with (3.4) inserted into (3.3):

$$(\rho v)_{x_{sh-}} - (\rho v)_{x_{sh+}} = (\rho_- - \rho_+)\frac{dx_{sh}}{dt} \tag{3.5}$$

or with $v_{sh} = \frac{dx_{sh}}{dt}$:

$$[\rho(v - v_{sh})]_+ = [\rho(v - v_{sh})]_-. \tag{3.6}$$

The relation (3.6) is one of $Rankine - Hugoniot's\ relations$. The remaining of these relations are obtained from the momentum and energy balance at a discontinuity.

Another way of writing (3.6) is:

$$v_{sh} = \frac{(\rho v)_+ - (\rho v)_-}{\rho_+ - \rho_-}. \tag{3.7}$$

Now we assume that the discontinuity is weak, which means that

$$\frac{|\rho_+ - \rho_-|}{\rho_+} << 1. \tag{3.8}$$

A Taylor expansion of v_{sh} gives by use of (3.1):

$$v_{sh} = \frac{1}{\rho_+ - \rho_-}\{(\rho_+ - \rho_-)j'(\rho_-) + \frac{1}{2}(\rho_+ - \rho_-)^2 j''(\rho_-) + ...\}. \tag{3.9}$$

A Taylor expansion of the propagation velocity $\tilde{c}(\rho) = j'(\rho)$ gives

$$\tilde{c}(\rho_+) = \tilde{c}(\rho_-) + (\rho_+ - \rho_-)j''(\rho_-) + ... \tag{3.10}$$

Comparison between (3.9) and (3.10) gives:

$$v_{sh} = \tilde{c}(\rho_-) + \frac{1}{2}[\tilde{c}(\rho_+) - \tilde{c}(\rho_-)] + O((\frac{\rho_+ - \rho_-}{\rho_+})^2) \tag{3.11}$$

or, with neglect of higher order contributions,

$$v_{sh} = \frac{1}{2}[\tilde{c}(\rho_+) + \tilde{c}(\rho_-)] \tag{3.12}$$

$$= c_0 + \frac{\beta}{2}(v_+ + v_-) \tag{3.13}$$

$$= c_0[1 + \frac{\beta}{2\rho_0}[(\rho_+ - \rho_0) + (\rho_- - \rho_0)]. \tag{3.14}$$

In the weak shock approximation founded on the condition (3.8) thus the velocity of the discontinuity is the mean value of the propagation velocities on both sides of the discontinuity.

The result (3.12) will now be used for handling the discontinuity in the solution of the nonlinear equation of acoustic waves (2.32). Using the solution (2.34) we call the propagation velocities in the fluid behind the discontinuity (or to the left of the discontinuity) and in the front (or to the right) of the discontinuity $f(\xi_-)$ and $f(\xi_+)$ respectively. The velocity of the discontinuity then becomes according to (3.12) and (2.32):

$$v_{sh} = c_0 + \frac{\beta}{2}[f(\xi_+) + f(\xi_-)]. \tag{3.15}$$

The position of the discontinuity is given by (2.35) with ξ put equal to either ξ_+ or ξ_- (the result shall be the same):

$$x_{sh} = \xi_+ + [c_0 + \beta f(\xi_+)]t = \xi_- + [c_0 + \beta f(\xi_-)]t \tag{3.16}$$

$$= \xi_+ + c_0[1 + \frac{\beta}{\rho_0}(\rho_+ - \rho_0)]t = \xi_- + c_0[1 + \frac{\beta}{\rho_0}(\rho_- - \rho_0)]t. \tag{3.17}$$

In the solution (2.34), (2.35) we cannot use ξ-values between ξ_- and ξ_+ in the curve describing the dependence of v on x. We will now find a simple method for constructing the position of x_{sh}. We will show that the position of x_{sh} is such that the areas of the two shadowed regions in Fig. 5 are equal. The total shadowed region $A(t)$ with sign is:

$$A(t) = \int \rho dx = \int_{\xi_-(t)}^{\xi_+(t)} \rho \frac{\partial x}{\partial \xi} d\xi \tag{3.18}$$

$$= \int_{\xi_-(t)}^{\xi_+(t)} \rho \frac{\partial}{\partial \xi}(x - x_{sh}) d\xi \tag{3.19}$$

$$= -\int_{\xi_-(t)}^{\xi_+(t)} [x(\xi, t) - x_{sh}(t)] \frac{d\rho(\xi)}{d\xi} d\xi. \tag{3.20}$$

Here we have integrated by parts. Note that $x(\xi_-, t)$ and $x(\xi_+, t)$ both are equal to x_{sh} so the integrand in (3.18) vanishes both in the lower and in the upper limit. The derivative $\frac{dA}{dt}$ then becomes:

$$\frac{dA}{dt} = -\int_{\xi_-(t)}^{\xi_+(t)} [\frac{\partial x(\xi, t)}{\partial t} - \frac{dx_{sh}}{dt}(t)] \frac{d\rho(\xi)}{d\xi} d\xi. \tag{3.21}$$

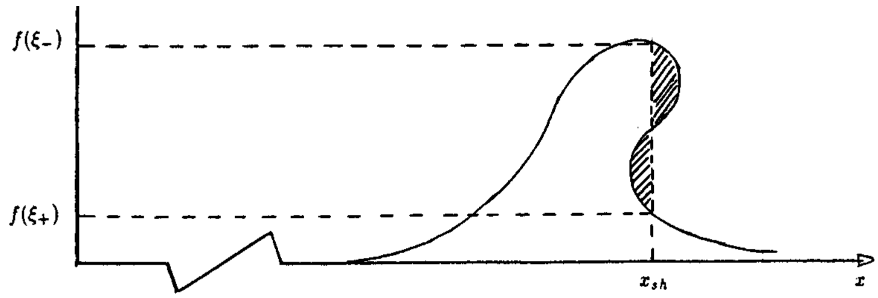


FIGURE 5. Dependence of v on x at time t

Using (2.35) in (3.21) we obtain

$$\begin{aligned}\frac{dA}{dt} &= -\int_{\xi_-(t)}^{\xi_+(t)} [\tilde{c}(\xi) - v_{sh}] \frac{d\rho(\xi)}{d\xi} d\xi \\ &= -\int_{\xi_-(t)}^{\xi_+(t)} \frac{\partial}{\partial \xi} \{(\rho v)(\xi) - v_{sh}(t)\rho(\xi)\} d\xi \\ &= -[(\rho v)(\xi_+) - (\rho v)(\xi_+)] + v_{sh}(t)[\rho(\xi_+) - \rho(\xi_-)] = 0. \qquad (3.22)\end{aligned}$$

That the result is zero follows from the expression for the shock velocity.

$$\frac{dA}{dt} = 0. \qquad (3.23)$$

At the moment when the discontinuity is created $A = 0$. Because of (3.23) A equals zero henceforth and the *rule of equal areas* is verified. Note that the area A is nothing but the mass of the part of the curve that is cut away. Hence we are cutting away a part with vanishing mass, which is necessary for mass conservation.

Let us now instead tackle the original expression. The rule of equal areas can be formulated in another way by making a partial integration of (3.18), where we now put $A(t) = 0$:

$$A = 0 = \int_{\xi_-}^{\xi_+} \frac{\partial x(\xi,t)}{\partial \xi} \rho(\xi) d\xi. \qquad (3.24)$$

Using (2.35) we obtain

$$\frac{\partial x(\xi,t)}{\partial \xi} = 1 + \frac{\beta c_0}{\rho_0} \frac{d\rho}{d\xi} t. \qquad (3.25)$$

Using this in (3.24) gives

$$0 = -\int_{\xi_-}^{\xi_+} [1 + \frac{\beta c_0}{\rho_0} \frac{d\rho}{d\xi} t] \rho(\xi) d\xi. \qquad (3.26)$$

Integration of the second term on the righthand side of (3.26) gives

$$\int_{\xi_-}^{\xi_+} \rho(\xi) d\xi = \frac{\beta c_0}{2\rho_0} [\rho^2(\xi_-) - \rho^2(\xi_+)] t. \qquad (3.27)$$

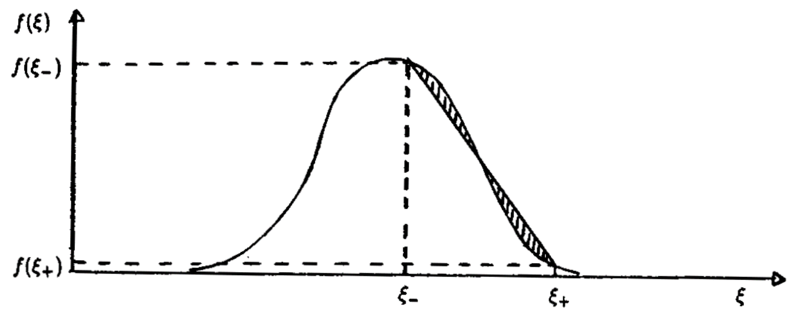


FIGURE 6. The rule of equal areas

From (3.16) we have

$$\xi_+ - \xi_- = \frac{\beta t c_0}{\rho_0}[\rho(\xi_-) - \rho(\xi_+)]. \tag{3.28}$$

Insertion of (3.28) into (3.27) gives a new formulation of the rule of equal areas, illustrated in Fig. 6:

$$\int_{\xi_-}^{\xi_+} \rho(\xi) d\xi = \frac{1}{2}(\xi_+ - \xi_-)[\rho(\xi_+) + \rho(\xi_-)]. \tag{3.29}$$

Triangular pulse

We consider a triangular pulse, which at $t = 0$ is given by

$$v = \{ \begin{array}{c} 0,\ x \leq -a \\ v_0 \frac{x+a}{a},\ -a \leq x \leq 0 \\ v_0 \frac{a-x}{a},\ 0 \leq x \leq a \\ 0,\ x \geq a \end{array} .$$

We are using a frame moving with the background speed c_0 so that the equation for v is

$$v_t + \beta v v_x = 0.$$

We want to find the location of the shock as a function of t as well as its strength.First introduce dimensionless variables

$$\begin{aligned} v &= v_0 v^*, \\ x &= a x^*, \\ t &= \frac{a}{v_0} t^* \end{aligned}$$

The equation is the same in the dimensionsless variables. We now skip the stars. The initial conditions now become (we denote x by ξ for $t = 0$)

$$v = \{ \begin{array}{c} 0,\ x \leq -1 \\ 1 + x,\ -1 \leq x \leq 0 \\ 1 - x,\ 0 \leq x \leq 1 \\ 0,\ x \geq 1 \end{array} .$$

In the picture we draw a line having $dv/d\xi = -1/\beta t$ obeying the equal area rule.

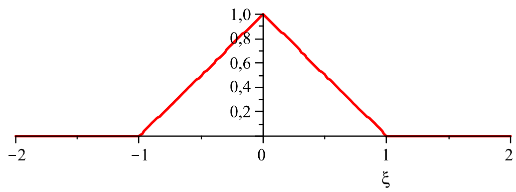


FIGURE 7. Original triangular pulse

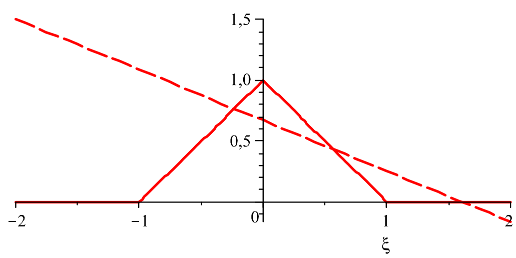


FIGURE 8. The triangular pulse with a line with inclination $-1/\beta t$.

The line cuts the triangle in ξ_- and ξ_+. Clearly $-1 \leq \xi_- \leq 0$ and $v_- = 1+\xi_-$. $1 \leq \xi_+$ and $v_+ = 0$. Then the area of the pulse between ξ_- and ξ_+ is

$$\int_{\xi_-}^{\xi_+} v d\xi = \int_{\xi_-}^{0} v d\xi + \int_{0}^{1} v d\xi = \frac{1}{2}(0 - \xi_-)(v_- + 1) + \frac{1}{2} = \frac{1}{2}[-\xi_-)(2 + \xi_-) + 1].$$

The area under the straight line between ξ_- and ξ_+ is

$$\frac{1}{2}(\xi_+ - \xi_-)v_- = \frac{1}{2}(\xi_+ - \xi_-)(1 + \xi_-)$$

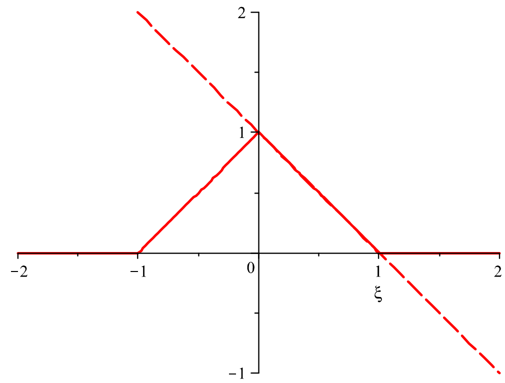


FIGURE 9. Appearance of shock

Equal areas means that these are equal

$$-\xi_-(2+\xi_-)+1=(\xi_+-\xi_-)(1+\xi_-)$$

Further, the position of the shock is given by ξ_- as well as ξ_+

$$x=\xi_-+\beta v_- t=\xi_++\beta v_+ t.$$

As v_+ vanishes, we obtain

$$\xi_+=\xi_-+\beta(1+\xi_-)t.$$

We insert this into the equation of equal area and obtain

$$\begin{aligned}(1+\beta t)(\xi_-^2+2\xi_-) &= 1-\beta t,\\ \xi_- &= -1+\sqrt{\frac{2}{1+\beta t}}\end{aligned}$$

We have that $-1\le\xi_-\le 0$, so the positive sign has to be taken and $\beta t\ge 1$. This means that the shock appears when $\beta t=1$,

$$t_{sh}=\frac{1}{\beta}.$$

This also gives

$$\begin{aligned}\xi_+ &= \beta t+(1+\beta t)\xi_-\\ &= -1+\sqrt{\frac{2}{1+\beta t}}\\ &\quad +\beta t\sqrt{\frac{2}{1+\beta t}}\\ &= -1+\sqrt{2(1+\beta t)}.\end{aligned}$$

The position of the shock at time t is (we now reinsert the stars)

$$x^*_{sh}(t)=\xi^*_+=-1+\sqrt{2(1+\beta t^*)}.$$

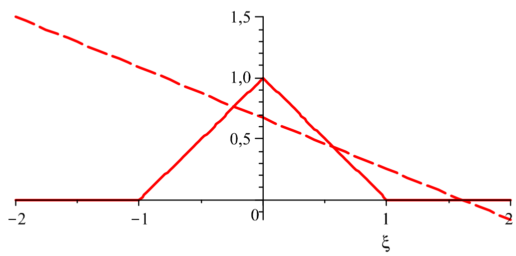


FIGURE 10. At $t = 2$.

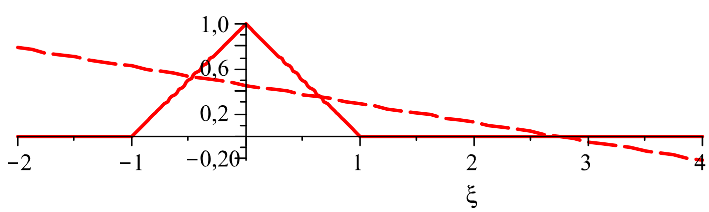


FIGURE 11. At $t = 5$.

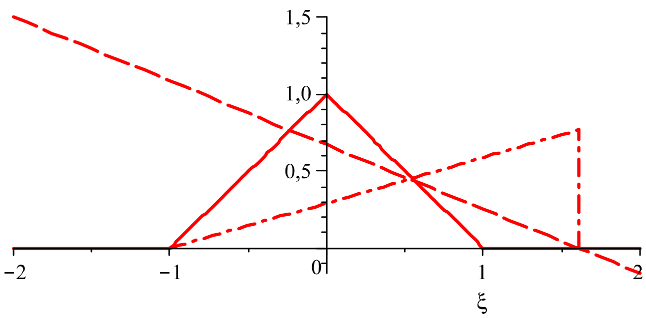


FIGURE 12. The dash dot line is the actual shape of the wave at $t = 3$.

Or with dimensions

$$x_{sh}(t) = [-1 + \sqrt{2(1 + \frac{\beta v_0 t}{a})}]a. \tag{3.30}$$

Its amplitude is

$$v_-^* = 1 + \xi_-^* = \sqrt{\frac{2}{1 + \beta t^*}}$$

and in dimensional units

$$v_- = v_0 \sqrt{\frac{2a}{a + \beta v_0 t}} \tag{3.31}$$

For large t we have

$$x_{sh}(t) \approx \sqrt{2\beta v_0 t a}. \tag{3.32}$$

For large t

$$v_- \approx v_0 \sqrt{\frac{2a}{\beta v_0 t}} \tag{3.33}$$

We finally also draw the real shape at $t = 3$ in fig. (12).

4. Traffic flow

In the general nonlinear first order partial differential equation (1.3)

$$\frac{\partial \rho}{\partial t} + \tilde{c}(\rho) \frac{\partial \rho}{\partial x} = 0 \tag{4.1}$$

the dependent variable ρ has the meaning of density. The velocity $\tilde{c}(\rho)$ is, as is found from the considerations of the equations (1.6)-(1.9), the velocity of an observer observing constant density ρ. We also introduce a flow velocity $j(\rho)$ and a velocity $V(\rho)$ of the medium, connected by the relation (j is the current):

$$j(\rho) = \rho V(\rho). \tag{4.2}$$

The continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{4.3}$$

must be compatible with (4.1), which gives

$$\tilde{c}(\rho) = j'(\rho). \tag{4.4}$$

The nonlinear wave theory based on (4.1) will now be applied to traffic flow. It is clear that the vehicle velocity $V(\rho) = \frac{j(\rho)}{\rho}$ is a decreasing function of ρ, starting at some finite value at $\rho = 0$ and reaching zero at $\rho = \rho_j$, where ρ_j is the maximal density of the vehicles, attained when they are packed along the road without interspace. The flow velocity $j(\rho)$, which means the number of passing vehicles per time unit, thus is zero both for $\rho = 0$ and for $\rho = \rho_j$. A maximum value q_m of $j(\rho)$ is attained for $\rho = \rho_m$, with $0 < \rho_m < \rho_j$. By observing the vehicles on a road with one drive the values of ρ_j, ρ_m and q_m are found. American conditions give $\rho_j =$ 225/mile, $\rho_m = 80$/mile, $q_m = 1500$/hour. The velocity of the cars which gives the maximum flow velocity thus is $v_m = \frac{q_m}{\rho_m} = 19$ miles/hour. At higher velocity the vehicles need so long distance between them that the number of passing vehicles per time unit decreases.

The propagation velocity of waves in the traffic flow is

$$\tilde{c}(\rho) = j'(\rho) = V(\rho) + \rho V'(\rho). \tag{4.5}$$

Since $V(\rho)$ is a decreasing function of ρ, $V'(\rho)$ is negative and from (4.5) we then conclude that the wave velocity is lower than the vehicle velocity. This means that waves propagate backwards through the traffic flow and drivers are warned by disturbances in front of them. The wave propagation velocity $\tilde{c}$ is the slope of the curve of $j(\rho)$ in Fig. 4 and thus a decreasing function of ρ. Therefore the waves move in the forward or backward direction with respect to the road, depending on whether $\rho < \rho_m$ or $\rho > \rho_m$. At the maximal flow velocity $j = q_m$ the wave velocity is zero with respect to the road and the wave velocity with respect to the vehicles is the same as the vehicle velocity $\frac{q_m}{\rho_m} \approx 19$ miles/hour.

We will now see if we can construct a model, which gives us a functional dependence $j(\rho)$. Assume that a driver has a reaction time δ, before he reacts upon a change in front of him. In this case the distance of security between the cars must be δV. If h is the distance between the fronts of two successive cars and L is the typical vehicle length, then

$$\delta V = h - L \tag{4.6}$$

or

$$V = \frac{h - L}{\delta}. \tag{4.7}$$

Since

$$h = \frac{1}{\rho} \tag{4.8}$$

and

$$L = \frac{1}{\rho_j} \tag{4.9}$$

we obtain from (4.7)

$$V(\rho) = \frac{\frac{1}{\rho} - \frac{1}{\rho_j}}{\delta} \tag{4.10}$$

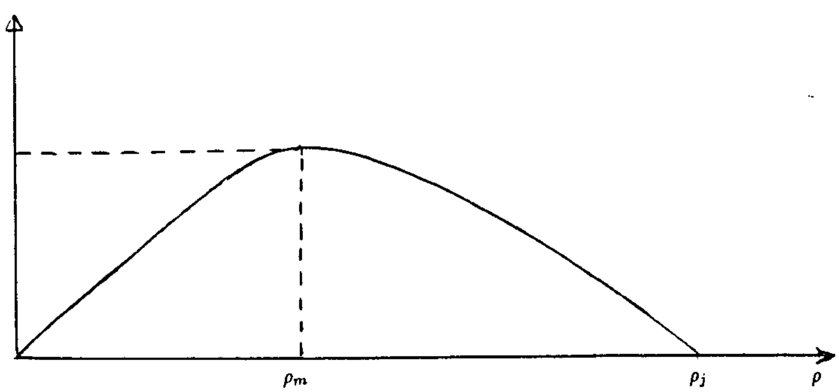
and then from (4.2)

$$j = \frac{1}{\rho_j \delta}(\rho_j - \rho). \tag{4.11}$$

However, the equation (4.11) only gives a realistic picture of the behaviour of the drivers in the neighborhood of $\rho = \rho_j$. A functional dependence $j(\rho)$, which agrees with observations in a wider range of ρ-values , is

$$j(\rho) = a\rho \log \frac{\rho_j}{\rho} = a\rho(\frac{\rho_j - \rho}{\rho}) - a\rho(\frac{\rho_j - \rho}{\rho})^3 + ... \tag{4.12}$$

Comparison



The flow velocity Q as a function of the density ρ in traffic flow

of the first term in the series expansion in (4.12) with (4.11) gives

$$a = \frac{1}{\rho_j \delta}. \tag{4.13}$$

>From (4.12) we then obtain

$$V - c = \frac{j(\rho)}{\rho} - j'(\rho) = a \log \frac{\rho_j}{\rho} - a \log \frac{\rho_j}{\rho} + a = a. \tag{4.14}$$

The logarithmic formula (4.12) does not give finite vehicle velocity for $\rho \to 0$. However, this failure of our model is of less importance, since the foundation of our model is weak in any case for very sparse traffic flow.

Because $j(\rho)$ is convex (see Fig. 8) with $j''(\rho) < 0$, then $c(\rho) = j'(\rho)$ is a decreasing function of ρ. This means that a local enhancement of density propagates so that a discontinuity is formed on the back side (see Fig. 9). Because, as we know from (4.5), individual cars move faster than the shockwaves, then a driver notices such a local density enhancement in front of him. Therefore he must brake suddenly, but he accelerates only slowly when he leaves the discontinuity point. This is seen

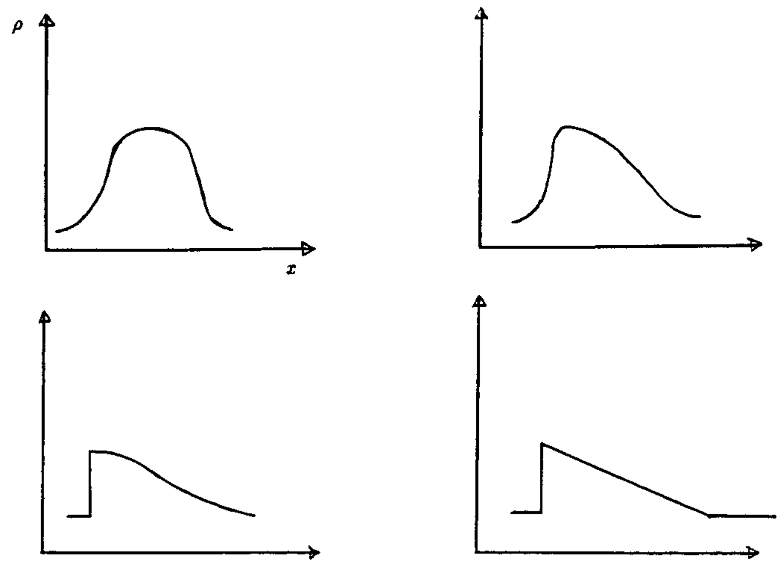


FIGURE 13. Discontinuity formation of a density enhancement in traffic flow

in Fig. 13. We will now discuss the discontinuity in the solution of the equation (4.1). As we have found in (3.7), the discontinuity travels with the velocity v_{sh}, where

$$v_{sh} = \frac{j(\rho_+) - j(\rho_-)}{\rho_+ - \rho_-}. \tag{4.15}$$

By means of the rule of equal areas we have already studied the discontinuity of the equation (2.32), which is a special case of (4.1). This special case is obtained by putting $j(\rho)$ equal to a second order polynomial in ρ, so that $\tilde{c}(\rho) = j'(\rho)$ becomes a first order polynomial in ρ. Under this assumption the rule of equal areas (3.29) can be used. The assumption that $j(\rho)$ is a second order polynomial in ρ is, as is seen from (4.12) or Fig. 4, not strictly true, but it includes the case of small disturbations around a value $\rho = \rho_0$. In this case we have approximately

$$j(\rho) = j(\rho_0) + j'(\rho_0)(\rho - \rho_0) + \frac{1}{2}j''(\rho_0)(\rho - \rho_0)^2 \tag{4.16}$$

or

$$j(\rho) = \alpha\rho^2 + \beta\rho + \gamma \tag{4.17}$$

and thus

$$\tilde{c}(\rho) = 2\alpha\rho + \beta. \tag{4.18}$$

The velocity of the discontinuity is given by (3.12):

$$v_{sh} = \frac{1}{2}[\tilde{c}(\rho_-) + \tilde{c}(\rho_+)] \tag{4.19}$$

with $c(\rho)$ given by (4.18).

By using (4.18) in (4.1) the position of the discontinuity can be determined by the rule of equal areas. If we choose to treat a differential equation for $\tilde{c}(\rho)$ instead of a differential equation for ρ, we can use the rule of equal areas independently of the form of the dependence $\tilde{c}(\rho)$. This can be seen by multiplying (4.1) with $\tilde{c}'(\rho)$:

$$\tilde{c}'(\rho)\rho_t + \tilde{c}(\rho)\tilde{c}'(\rho)\rho_x = 0 \tag{4.20}$$

or

$$\tilde{c}_t + \tilde{c}\tilde{c}_x = 0. \tag{4.21}$$

Thus in (4.21) we have obtained an equation possible to treat by the rule of equal areas independently of the dependence $j(\rho)$. Having solved (4.21) and obtained the discontinuity we can find ρ by inverting the known dependence $c(\rho)$.

A continuous solution of (4.21) can be written

$$\tilde{c} = F(\xi) \tag{4.22}$$

$$x = \xi + F(\xi)t, \tag{4.23}$$

where

$$F(x) = \tilde{c}(f(x)) \tag{4.24}$$

and $f(x)$ is given by the initial condition for the density of the traffic flow:

$$\rho(x, t = 0) = f(x). \tag{4.25}$$

The rule of equal areas can be written according to (3.29):

$$\frac{1}{2}\{F(\xi_-) + F(\xi_+)\}(\xi_+ - \xi_-) = \int_{\xi_-}^{\xi_+} F(\xi)d\xi. \tag{4.26}$$

If the position of the discontinuity is $x = s(t)$ at the time t, we have in analogy with (3.16):

$$s(t) = \xi_- + F(\xi_-)t = \xi_+ + F(\xi_+)t. \tag{4.27}$$

We now assume that the initial flow fulfils the condition

$$\begin{array}{rcl} \tilde{c}(x, t = 0) & = & F(x) \\ F(\xi) & < & c_0,\ -l < \xi < 0 \\ F(\xi) & = & c_0,\ \xi < -l,\ \xi > 0. \end{array} \tag{4.28}$$

At $t = 0$ we thus have a local reduction of the wave velocity. Because, according to Fig. 8, the wave velocity $c(\rho)$ in traffic flow is a decreasing function of the density ρ, the reduction of wave velocity means an increase of density. The rule of equal areas (4.26) can now be written:

$$\frac{1}{2}\{2c_0 - F(\xi_-) - F(\xi_+)\}(\xi_+ - \xi_-) = \int_{\xi_-}^{\xi_+} [c_0 - F(\xi)]d\xi. \tag{4.29}$$

At the time $t = t_{sh}$, at which the discontinuity sets in, ξ_+ and ξ_- have one and the same value, situated between $-l$ and 0. For t increasing still more, ξ_+ increases and ξ_- decreases. In Fig. 14 the initial condition is pictured and in Fig. 15 the rule of equal areas for $\tilde{c}$ as a function of ξ and x respectively.>From the left picture in Fig. 15 we see that the straight line between the points $(\xi_-, F(\xi_-))$ and $\xi_+, F(\xi_+))$ becomes more horizontal for growing t values. At the same time ξ_- decreases and

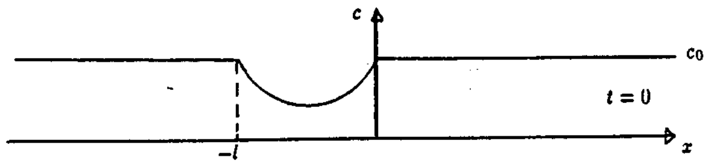


FIGURE 14. Initial condition for shock in traffic flow

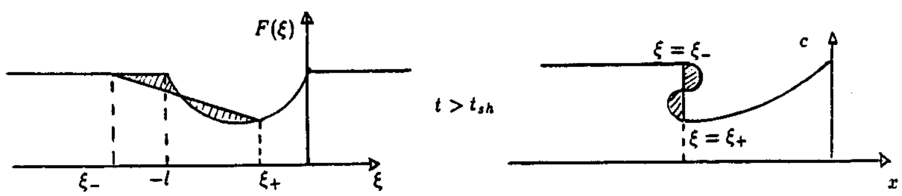


FIGURE 15. The rule of equal areas in traffic flow

becomes less than $-l$ and ξ_+ increases and approaches zero. If we assume that $\xi_- < -l$, we obtain from (4.29):

$$\frac{1}{2}\{c_0 - F(\xi_+)\}(\xi_+ - \xi_-) = \int_{-l}^{\xi_+} [c_0 - F(\xi)]d\xi. \tag{4.30}$$

For $\xi_- < l$ we obtain from (4.27):

$$t = \frac{\xi_+ - \xi_-}{F(\xi_-) - F(\xi_+)} = \frac{\xi_+ - \xi_-}{c_0 - F(\ xi_+)}. \tag{4.31}$$

Then we obtain from (4.30) and (4.31):

$$\frac{1}{2}\{c_0 - F(\xi_+)\}^2 t = \int_{-l}^{\xi_+} [c_0 - F(\xi)]d\xi. \tag{4.32}$$

The position of the discontinuity $s(t)$ and the value c_+ of c just to the right of the discontinuity are determined by the relations

$$s(t) = \xi_+ + F(\xi_+)t \tag{4.33}$$

$$c_+ = F(\xi_+), \tag{4.34}$$

where ξ_+ is given by (4.32).

When t approaches infinity ξ_+ approaches zero and $F(\xi_+)$ approaches c_0. Then we obtain from (4.32):

$$F(\xi_+) \approx c_0 - \surd(\frac{2A}{t}),\ t \to \infty, \tag{4.35}$$

where

$$A = \int_{-l}^{0} [c_0 - F(\xi)]d\xi. \tag{4.36}$$

Thus A is the area of the initial reduction of c below its undisturbed value c_0. Using (4.35) and the fact that $\xi_+ \to 0$ when $t \to \infty$ we find the asymptotic forms of (4.33) and (4.34):

$$s(t) \approx c_0 t - \surd(2At),\ t \to \infty \tag{4.37}$$

$$c_+ - c_0 = -\surd(\frac{2A}{t}). \tag{4.38}$$

The solution valid to the right of the discontinuity at $x = s(t)$ is given by (4.22), (4.23) for $\xi_+ < \xi < 0$. Because, for $t \to \infty$ and $\xi_+ \to 0$, the ξ-values in the interval $\xi_+ < \xi < 0$ tend to zero, we find from (4.22), (4.23):

$$x = ct \tag{4.39}$$

or

$$c(x,t) = \frac{x}{t}. \tag{4.40}$$

For a fixed t value the straight line solution (4.40) with the slope $\frac{1}{t}$ is valid in the x interval

$$s(t) < x < c_0 t, \tag{4.41}$$

where the upper limit is determined by the fact that the front of the depression propagates with the undisturbed velocity c_0. For $t \to \infty$ we obtain from (4.41) using (4.37):

$$c_0 t - \surd(2At) < x < c_0 t. \tag{4.42}$$

The limit form of the wave velocity reduction in the interval (4.42) is pictured in Fig. 12.

>From Fig. 16 we see that all the structure of the initial wave velocity reduction is lost in its limit form for $t \to \infty$. The just completed analysis of the discontinuity in the wave propagation velocity c can be applied immediately to the discontinuity in the density ρ. In the asymptotic region $t \to \infty$ we can assume that $\rho - \rho_0$ is so small that the relation

$$\tilde{c} - c_0 = \tilde{c}'(\rho_0)(\rho - \rho_0),\ \tilde{c}'(\rho_0) < 0 \tag{4.43}$$

is valid independently of the form of the function $j(\rho)$. As we know from (4.17), the equation (4.43) is exact only for the case that $j(\rho)$ is a second degree polynomial in ρ. We now start with a density increase, so that

$$\begin{aligned} \rho(x,0) &= f(x),\ -l < x < 0 \\ \rho(x,0) &= \rho_0,\ x < -l,\ x > 0. \end{aligned} \tag{4.44}$$

>From (4.40), (4.42) and (4.43) we obtain immediately

$$\rho - \rho_0 = \frac{x - c_0 t}{\tilde{c}'(\rho_0)t},\ c_0 t - \surd(2Bt) < x < c_0 t, \tag{4.45}$$

where B is the area earlier called A and now written:

$$B = |\tilde{c}'(\rho_0)| \int_{-l}^{0} [f(\xi) - \rho_0] d\xi. \tag{4.46}$$

At the same position there is thus a discontinuous increase of wave propagation velocity density and a discontinuous increase of vehicle density. From (4.5) we then conclude (since $V'(\rho) < 0$) that the vehicle velocity has a discontinuous decrease at

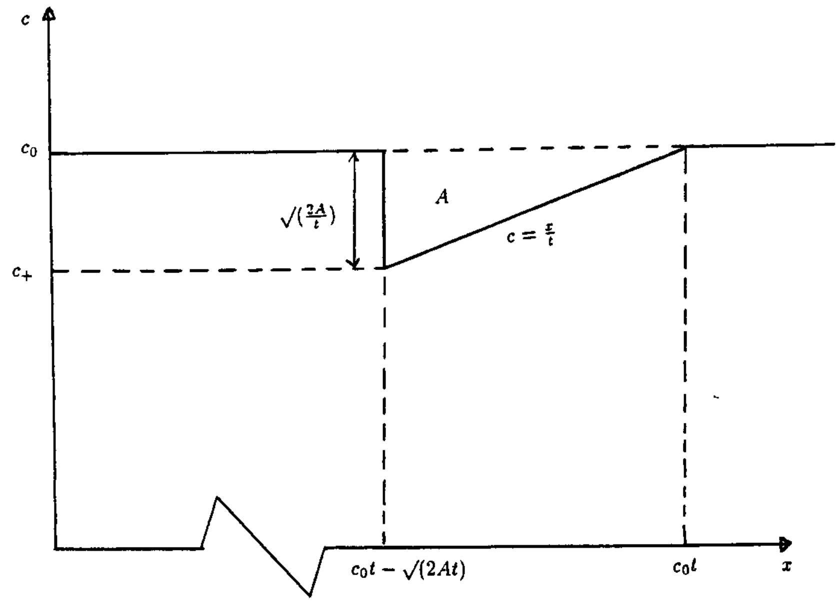


FIGURE 16. Limit form of a wave velocity depression in traffic flow

that position. To the right of the discontinuity the velocities and the density recover the values they had to the left of the discontinuity. The recovering is completed in a distance growing as $\sqrt{t}$, while simultaneously the discontinuity is decaying as $\frac{1}{\sqrt{t}}$.

THE END