Homework 5, due 10/3
5C1212, Mekanik/NADA

Finite Element Methods for the Stokes Equations

We will study the Stokes equations in the form

$$\begin{aligned}
-\Delta \boldsymbol{u} + \nabla p &= 0 && \text{in } \Omega, \\
\nabla \cdot \boldsymbol{u} &= 0 && \text{in } \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_0, \\
\boldsymbol{u} &= \boldsymbol{g} && \text{on } \Gamma_1, \\
-\frac{\partial \boldsymbol{u}}{\partial n} + \boldsymbol{n} p &= 0 && \text{on } \Gamma_2.
\end{aligned} \tag{1}$$

where Ω is an open, bounded, and connected region in the plane with $\boldsymbol{n}$ being the outward-directed unit normal on the boundary. The velocity and pressure fields are denoted $\boldsymbol{u} = (u, v)$ and p. Inhomogeneous and homogeneous Dirichlet boundary conditions are supplied on parts Γ_0 and Γ_1 of the boundary, respectively. The boundary condition on Γ_2 is of dubious physical relevance in itself, but can be used as an artificial outflow boundary condition.

We will apply the Finite-Element method to solve the boundary-value problem (1) numerically.

A mixed finite element approximation of equation (1) may be written

$$\begin{aligned}
&\text{Find } \boldsymbol{u}_h \in \boldsymbol{V}^h \text{ and } p_h \in H^h \text{ such that} \\
&\int_\Omega \nabla \boldsymbol{u}_h \cdot \nabla \boldsymbol{w}_h \, d\Omega - \int_\Omega p_h \nabla \cdot \boldsymbol{w}_h \, d\Omega = 0, \qquad \forall \boldsymbol{w}_h \in \boldsymbol{V}_0^h \\
&\int_\Omega q_h \nabla \cdot \boldsymbol{u}_h \, d\Omega = 0, \qquad \forall q_h \in H^h.
\end{aligned} \tag{2}$$

The space of pressures, H^h, should be a subspace of square-integrable functions on Ω, and the spaces of the velocities, $\boldsymbol{V}^h$ and $\boldsymbol{V}_0^h$, should be subspaces of functions with square-integrable gradients. Note that the velocity solution $\boldsymbol{u}_h$ is defined in a different space than the test function $\boldsymbol{w}_h$. Both the functions in $\boldsymbol{V}^h$ and $\boldsymbol{V}_0^h$ vanish on boundary Γ_0. However, functions in the test-function space $\boldsymbol{V}_0^h$ should vanish also on Γ_1, whereas functions in the solution-space $\boldsymbol{V}^h$ should satisfy (approximations of) the inhomogeneous boundary conditions on Γ_1.

We use continuous, piecewise-linear functions on triangular meshes both for the velocity and pressure approximations. However, different meshes will be allowed for the velocity and pressure approximations to assure that the LBB condition is satisfied.

The complication associated with different spaces for the velocity solution and test function can be handled by splitting up $\boldsymbol{u}_h$ in one function $\boldsymbol{g}_h$ satisfying the boundary conditions on Γ_1 and one function $\boldsymbol{u}_{h,0}$ being zero on all Dirichlet boundaries,

$$\boldsymbol{u}_h = \boldsymbol{g}_h + \boldsymbol{u}_{h,0}. \tag{3}$$

Here, $\boldsymbol{g}_h$ is a function in $\boldsymbol{V}^h$ agreeing with the boundary conditions at the nodes of Γ_1 and being zero at all other nodes in the mesh. This means that $\boldsymbol{u}_{h,0} \in \boldsymbol{V}_0^h$, so the unknown function and the test functions belong to the same space.

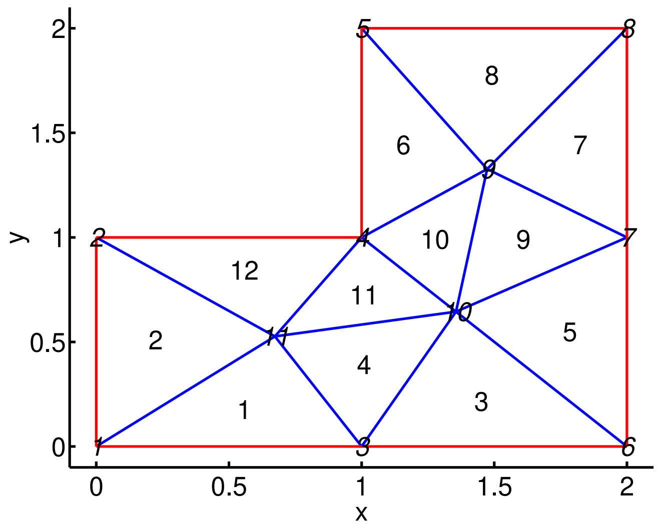

Figure 1: An example mesh. The mesh-point numbering is indicated with italics. The triangle ordering is indicated with upright numbers.

Question 1.

Using equation (2), show that

$$\begin{aligned}
&\text{if } \boldsymbol{u}_{h,0} \in \boldsymbol{V}_0^h \text{ and } p_h \in H^h \text{ then} \\
&\int_\Omega \nabla \boldsymbol{u}_{h,0} \cdot \nabla \boldsymbol{w}_h \, d\Omega - \int_\Omega p_h \nabla \cdot \boldsymbol{w}_h \, d\Omega = -\int_\Omega \nabla \boldsymbol{g}_h \cdot \nabla \boldsymbol{w}_h \, d\Omega, \qquad \forall \boldsymbol{w}_h \in \boldsymbol{V}_0^h \\
&\qquad\qquad \int_\Omega q_h \nabla \cdot \boldsymbol{u}_{h,0} \, d\Omega = -\int_\Omega q_h \nabla \cdot \boldsymbol{g}_h \, d\Omega, \qquad \forall q_h \in H^h.
\end{aligned} \tag{4}$$

The solution $\boldsymbol{u}_h$ is then obtained from expression (3).

The finite-element approximation (4) yields the linear system

$$\begin{pmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{N} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_b \end{pmatrix} \tag{5}$$

for the nodal values of the velocity and pressure unknowns $\mathbf{u}$ and $\mathbf{p}$. The matrices $\mathbf{K}$, $\mathbf{G}$, and $\mathbf{G}^T$ are discrete counterparts to the (negative) vector Laplacian, the gradient, and the divergence operators. The matrix $\mathbf{N}$ is the zero matrix for approximation (4). The right-hand side vectors come from the inhomogeneous Dirichlet conditions on Γ_1.

The velocity unknowns $\mathbf{u}$ is a column vector of the *unknown* nodal values of $\boldsymbol{u}_{h,0}$ in equation (4). Thus, if N_u is the number of node points in the strict interior of the domain and on Γ_2, the dimension of $\mathbf{u}$ is $2N_u$; the first N_u elements of $\mathbf{u}$ are associated with the x-component of $\boldsymbol{u}_{h,0}$ and the rest with the y-component. Note that the nodal points on Γ_0 and Γ_1 are not included since $\boldsymbol{u}_{h,0}$ vanishes there. Assume that the mesh in figure 1 is a velocity mesh, and that the rightmost vertical boundary is Γ_2. In this case, we have $N_u = 6$, and the velocity unknowns in the vector $\mathbf{u}$ are nodal values at mesh nodes 6, 7, 8, 9, 10, and 11, that is,

$$\mathbf{u} = (u_6, u_7, u_8, u_9, u_{10}, u_{11}, v_6, v_7, v_8, v_9, v_{10}, v_{11}).$$

The pressure is unknown in the interior as well as on the boundary. Thus, the dimension of column vector $\mathbf{p}$ is equal to the total number of node points in the pressure mesh, N_p.

Question 2.

Assume that the mesh in figure 1 is also the corresponding pressure mesh (note that if the matrix $\mathbf{N}$ is chosen appropriately this discretization is stable). Assume also that the nodal points Γ_0 is the empty set. What is the dimension of the discrete gradient matrix $\mathbf{G}$ and the vector Laplacian $\mathbf{K}$ which has the structure

$$\mathbf{K} = \begin{pmatrix} \widehat{\mathbf{K}} & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbf{K}} \end{pmatrix},$$

where $\widehat{\mathbf{K}}$ is the scalar discrete (negative) Laplacian.

Also show that the right-hand side vectors have the structure

$$\mathbf{f}_u = -\mathbf{K}_{\Gamma_1}\mathbf{g} = -\begin{pmatrix} \widehat{\mathbf{K}}_{\Gamma_1} & 0 \\ 0 & \widehat{\mathbf{K}}_{\Gamma_1} \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{g}}_1 \\ \widehat{\mathbf{g}}_2 \end{pmatrix},$$

$$\mathbf{f}_p = -\mathbf{G}_{\Gamma_1}^T\mathbf{g} = -\begin{pmatrix} \widehat{\mathbf{G}}_{\Gamma_1}^T & \widehat{\mathbf{G}}_{\Gamma_1}^T \end{pmatrix} \begin{pmatrix} \widehat{\mathbf{g}}_1 \\ \widehat{\mathbf{g}}_2 \end{pmatrix};$$

and that the column vectors $\widehat{\mathbf{g}}_1$ and $\widehat{\mathbf{g}}_2$ contain nodal values of the x- and y-components of the boundary condition data $\boldsymbol{g}$ on Γ_1. What is the dimension of $\widehat{\mathbf{K}}_{\Gamma_1}$, $\widehat{\mathbf{G}}_{\Gamma_1}$, $\widehat{\mathbf{g}}_1$ and $\widehat{\mathbf{g}}_2$?