Study questions

1. Define and use the Rankine-Hugoniot jump condition to compute the shock speed for the following problem

$$\begin{aligned} u_t + uu_x &= 0 \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= \begin{cases} 1 & x \le 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

2. Define the entropy condition for a scalar conservation law.

$$u_t + f(u)_x = 0 \quad -\infty < x < \infty, \quad t > 0$$

   with a convex flux function $f(u)$. The shock is moving with speed s and the state to the left is given by u_L and the state to the right by u_R.

   Why do we need an entropy condition ?

3. Write a difference approximation (in 1D) on conservative form and define the notation.

4. Solve

$$u_t + (\frac{u^2}{2})_x = 0, \qquad u(x,0) = \begin{cases} 1 & x < 0 \\ 0 & x \ge 0 \end{cases} \tag{1}$$

   by the upwind scheme on the following form,

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x} u_j^n (u_j^n - u_{j-1}^n)$$

   a) Will the numerical solution converge to the analytical solution?

   b) If not, suggest another way of solving 1.

5. Define a total variation decreasing (TVD) method. Why is this a desirable property ?

6. Investigate the one-sided difference scheme

$$u_j^{n+1} = u_j^n - a\frac{\Delta t}{\Delta x}(u_j^n - u_{j-1}^n)$$

   for the advection equation

$$u_t + au_x = 0$$

   Consider the cases $a > 0$ and $a < 0$.

   a) Prove that the scheme is consistent and find the order of accuracy. Assume k/h constant.

   b) Determine the stability requirement for $a > 0$ and show that it is unstable for $a < 0$.

7. Apply Lax-Friedrichs scheme to the linear wave equation

$$u_t + au_x = 0.$$

   a) Write down the modified equation.

   b) What type of equations is this?

   c) What kind of behavior can we expect from the solution?

8. A the three-point centered scheme applied to

$$u_t + au_x = 0, \quad a > 0.$$

   yields the approximation

$$u_j^{n+1} = u_j^n + \frac{a\Delta t}{2\Delta x}(u_{j+1} - u_{j-1})$$

   Show that this approximation is not stable even though the CFL condition is fulfilled.

9. What does Lax's equivalence theorem state?

10. What is the condition on the $n \times n$ real matrix $A(\mathbf{u})$ for the system

$$\mathbf{u}_t + A\mathbf{u}_x = 0$$

   to be hyperbolic ?

11. The barotropic gas dynamic equations

$$\begin{aligned} \rho_t + \rho u_x &= 0 \\ u_t + uu_x + \frac{1}{\rho}p_x &= 0 \end{aligned} \tag{2}$$

   where

$$p = p(\rho) = C\rho^\gamma$$

   i and C a constant, can be linearized by considering small perturbations (ρ', u') around a motionless gas.

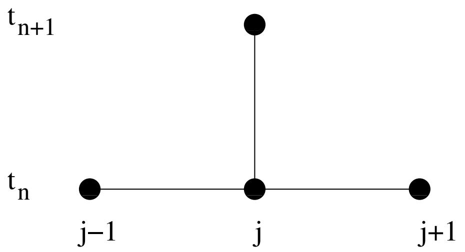
   a) Let $\rho = \rho_0 + \rho'$ and $u = u_0 + u'$ where $u_0 = 0$. Linearize the system (2) and show that this yields the following linear system (the primes has been dropped)

$$\begin{aligned} \rho_t + \rho_0 u_x &= 0 \\ u_t + \frac{a^2}{\rho_0}\rho_x &= 0 \end{aligned} \tag{3}$$

   where a is the speed of sound. a and ρ_0 are constants.

   b) Is the system given by (3) a hyperbolic system? Motivate your answer.

   c) Determine the characteristic variables in terms of ρ and u.



d) Determine the partial differential equations the characteristic variables fulfill - characteristic formulation.

e) Given initial conditions at $t = 0$ and let $-\infty < x < \infty$ (no boundaries)

$$\rho(0, x) = sin(x) \quad u(0, x) = 0$$

determine the analytical solution to (3) for $t > 0$. Hint: Start from the characteristic formulation.

12. The linearized form of (3) is given by

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & \rho_0 \\ a^2/\rho_0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} \rho \\ u \end{pmatrix}_x = 0, \tag{4}$$

where a is the speed of sound. a and ρ_0 are constants.

a) Draw the domain of dependence of the solution to the system (4) in a point P in the x-t plane.

b) The system is solved numerically on a grid given by $x_j = j\Delta x, j = 0, 1, 2...$ and $t_n = n\Delta t, n = 0, 1, 2, ...$ using an explicit three-point scheme, see the figure below.

Draw the domain of dependence of the numerical solution at P (in the same figure as a)) of the three-point scheme in the case when

i) the CFL condition is fulfilled

ii) the CFL condition is NOT fulfilled.

Assume that P is a grid point.

13. To solve Euler equations in 1D

$$\rho_t + \rho u_x + u\rho_x = 0$$
$$u_t + uu_x + \frac{1}{\rho}p_x = 0$$
$$p_t + \rho c^2 u_x + up_x = 0$$

How many boundary conditions must be added at (motivate your answer) inflow boundary when the flow is

a) Supersonic

b) Subsonic

outflow boundary when the flow is

c) Supersonic

d) Subsonic

14. Describe the ideas behind a flux splitting scheme for solving a non-linear hyperbolic system of equations,

$$\mathbf{U}_t + F(\mathbf{U})_x = 0$$

15. a) Show that a vector field w_i can be decomposed into

$$w_i = u_i + \frac{\partial p}{\partial x_i}$$

where u is is divergence free and parallel to the boundary.

b) Apply this to the Navier-Stokes equations, show that the pressure term disappears and recover an equation for the pressure from the gradient part.

16. From the differential form of the Navier-Stokes equations obtain

a) the Navier-Stokes equations in integral form used in finite-volume discretizations,

b) a variational form of the Navier-Stokes equations used in finite-element discretizations.

17. a) Write down the appropriate function spaces for the pressure and velocity used to define weak solutions to the Navier-Stokes equations.

b) Explain the concept of essential and natural boundary conditions.

18. a) How is a finite-element approximation defined?

b) Explain how to convert a FEM discretization to an algebraic problem, e.g. that the Navier-Stokes equations yield

$$\begin{pmatrix} \nu\mathbf{K} + \mathbf{C}(u) & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

19. a) By choosing $\mathbf{z}_h = \mathbf{u}_h$ in the finite element approximation of the Stokes problem, show that any Galerkin velocity solution is stable.

b) Describe and interpret the LBB condition.

20. Choice of elements. Discrete elements pairs

a) constant pressure-bilinear velocities,

b) the Taylor-Hood pair,

c) a stable choice with piecewise linear velocities.

21. a) Derive the finite volume (FV) discretization on arbitrary grids of the continuity equation ($\partial u_i/\partial x_i = 0$),

    b) derive the FV discretization for Laplace equation on a Cartesian grid,

    c) show that both are equivalent to a central difference approximation for Cartesian grids.

22. Derive the finite element (FEM) discretization for Laplace equation on a Cartesian grid and show that it is equivalent to a central difference approximation.

23. Iterative techniques for linear systems.

    a) Define Gauss-Seidel iterations for the Laplace equations,

    b) Define the 2-level multigrid method for the Laplace equation,

24. State the difficulties associated with the the finite-volume discretizations of the Navier-Stokes equations on a colocated grid? and show the form of the spurious solution which exist.

25. a) Define an appropriate staggered grid that can be used for the discretization of the Navier-Stokes equations,

    b) write down the FV discretization of the Navier-Stokes equations on a staggered cartesian grid,

    c) discuss how to treat noslip and inflow/outflow boundary conditions.

26. Time dependent flows.

    a) Define a simple projection method for the time dependent Navier-Stokes equations

$$\frac{d}{dt}\begin{pmatrix} u \\ 0 \end{pmatrix} + \begin{pmatrix} \mathbf{N}(u) & \mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{pmatrix}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

    b) show in detail the equation for the pressure to be solved at each time step and discuss the boundary conditions for the pressure.

27. Time step restriction for Navier-Stokes solutions.

    a) Motivate the use of an appropriate form of the advection-diffusion equation as a model equation for stability analysis,

    b) derive the time step restrictions for the 1D version of that equation,

    c) state the 2D equivalent of that restriction.